TIGHT BINGING BOOK

THEORY OF
VIBRATING SYSTEMS AND SOUND

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By

IRVING B. CRANDALL, PH.D.

Member of the Technical Staff

BELL TELEPHONE LABORATORIES, INC.

NEW YORK

D. VAN NOSTRAND COMPANY

EIGHT WARREN STREET

1926

Printed in the U. S. A.

PREFACE

THIS treatment of the Theory of Sound is intended for the student of physics who has given a certain amount of attention to analytical mechanics, and who desires a sufficient acquaintance with the theory of sound and its recent applications to bring it into balance with his studies in other branches of mechanical science. To this end a few fundamentals of theory are emphasized, principally for the purpose of introducing other (and less abstract) matters. With the background thus assumed on behalf of the reader, it does not seem necessary to include such developments as the proof of Lagrange's Equations, the theory of elastic deformations, or the theory of electrical networks, with which he will have become familiar through reading collateral literature; but on the other hand, there are (for example) certain analogies between the theories of mechanical and electrical oscillations which might well be indicated, since they have become so important a feature of acoustical technique. In a general way it is hoped, herein, to supplement rather than to replace the accepted treatises on Sound.

From the nature of things, there are many references in this text to the classical works of Lord Rayleigh and Prof. Horace Lamb. My obligations would be even greater, if (departing from the spirit of the text) I had attempted to treat anew the general theories of strings, bars, and plates, and the theory of Harmonic Analysis, which occupy so much space in the classical literature. The reader may take what he needs of these theories from the standard treatises. Coming to later sources, the published work of C. V. Drysdale on various problems in the mechanics of fluids has furnished much useful special information. It will also be recognized that Chapter V is to a degree dependent on the original studies in Architectural Acoustics recorded in W. C. Sabine's Collected Papers, as well as on the

more recent work of Dr. P. E. Sabine, which he has kindly placed at my disposal with many helpful suggestions. I take it for granted that the research worker in acoustics will supplement what he may find in the present volume by copious reading of these and other sources referred to in the text, and in Appendix B, which is intended as a guide to the newer literature.

In the recent years great progress has been made in applied acoustics, and a number of contributions have originated in Bell Telephone Laboratories. The studies on which this book is based were made entirely on this atmosphere, and the material was first presented by the writer in one of the "out-of-hour" courses in the Laboratories. Through these courses the members of its technical staff obtain from each other theoretical training and first-hand knowledge of new methods. It is to be expected that, from time to time, similar presentations of other subjects of importance in our work will appear in style uniform with the present volume.

Much of the present book has also received class presentation in a course given at the Massachusetts Institute of Technology during the spring of 1926 at the invitation of the Department of Electrical Engineering.

It is impossible to acknowledge in detail the helpful criticisms and contributions I have received from many of my associates; I am also indebted to others, in the academic sphere, for friendly suggestions on various matters in which they were specially interested. This cooperation has undoubtedly made possible a better book. In conclusion, I may express my thanks to the staff of the Technical Library of our Laboratories, whose services in furnishing literature and bibliographical data have been indispensable; and to Mr. L. A. MacColl, M.A., who has read all the proofs for correctness, both as to text and as to mathematical style.

IRVING B. CRANDALL.

NEW YORK,
June 1, 1926.

CONTENTS

CHAPTER I

Simple Vibrating Systems

CHAPTER II

General Theory of Vibrating Systems; Resonators and Filters

CHAPTER III

The Propagation of Sound

CHAPTER IV

RADIATION AND TRANSMISSION PROBLEMS

CHAPTER V

THE ACOUSTICS OF CLOSED SPACES; ABSORPTION, REFLECTION AND REVERBERATION

APPENDICES

LIST OF SYMBOLS

ξ, $\dot{\xi}$, $\ddot{\xi}$,	displacement, velocity, acceleration.
ξ_0, $\dot{\xi}_0$, $\ddot{\xi}_0$,	maximum values; e.g., $\xi = \xi_0 e^{i\omega t}$.
r, m, s,	resistance, mass and stiffness coefficients, in one degree of freedom.
Ψ_1, Ψ_k,	applied forces; e.g., $\Psi = \Psi_0 e^{i\omega t}$.
Z, Z_k, Z_{jk},	impedances; defined as needed.
f, ω,	frequency; $\omega = 2\pi f$.
f_0, f_k, n, n_k,	natural frequency; $n = 2\pi f_0$, $n_k = 2\pi f_k$.
Δ, Δ_k,	damping coefficient or inverse modulus of decay.
$n' = \sqrt{n^2 - \Delta^2}$,	frequency of natural oscillations $\times$ 2π.
λ, λ_k, λ_k',	roots of auxiliary equation.
a_{jk}, b_{jk}, c_{jk},	mass, resistance and stiffness coefficients in general theory.
T, V, F,	kinetic energy, potential energy and dissipation functions.
$\boldsymbol{D}$,	LAGRANGE's determinant of coefficients of equations of motion.
α_j, α_k; β_j, β_k;	factors of $\boldsymbol{D}$.
τ,	tension.
ρ,	density. (As applied to lines, surfaces or volumes.)
P, p_0,	total pressure, mean pressure.
p, δp,	excess pressure with respect to mean.
R, R_k,	radiation resistance.
R, R_1,	resistance coefficients per unit length, constricted conduits.
μ,	viscosity coefficient.
$\nu = \dfrac{\mu}{\rho}$,	kinematic viscosity.
γ,	ratio of specific heats for a gas.

κ, κ',	coefficient of cubic elasticity; κ' for a gas $= \gamma p_{\text{c}}$
s, Δ,	condensation and dilatation; $s = -\Delta$.
c, λ,	velocity of sound, wave length; $\lambda f = c$.
S, S_1,	area of equivalent piston; area of wall surfac
K, K_1,	conductivity of an orifice.
V_0, V,	volume or capacity of a resonator; volume a room.
A,	strength of small source of sound; $A = S\bar{\xi}_0$.
$k = \frac{\omega}{c}$,	phase constant in a non-absorbing medium.
$\beta = \frac{\omega}{c'}$,	phase constant in an absorbing medium.
c',	phase velocity; in general, $c' \neq c$.
α,	attenuation coefficient in wave transmission.
ξ, η, ζ,	fluid displacements parallel to x, y, z, axes.
ϕ,	velocity potential.
E,	energy density in medium.
$\frac{dW}{dt}$,	intensity. $\frac{dW}{dt} = E \cdot c$; $W = T + V$.
∇^2,	LAPLACE'S operator.
Ω,	solid angle.
r,	amplitude reflection coefficient.
$R = r^2$,	energy reflection coefficient.
t, t_{jk},	transmission coefficients.
A,	energy absorption coefficient. $(A + R = 1.)$
$a = \Sigma A_j S_j$,	total absorbing power of area ΣS_j.
T,	reverberation time.
K,	SABINE'S constant for reverberation.

CHAPTER I

Simple Vibrating Systems

1. *Introduction; the Principle of Superposition*

The physical bases of the Theory of Sound may be reduced to three primary phenomena. The first of these is that sound waves are produced whenever a vibrating body is placed in contact with an elastic substance. Next in order is the transmission of sound by the elastic substance or medium: the velocity of propagation being greater, the greater the ratio of stiffness of the medium to its density. Sound, in undergoing transmission, has all the characteristics of wave-motion. The parts of the medium which are traversed execute periodic motion; the volume elements in the medium undergo periodic expansion and contraction; or what is the same thing, periodic changes in density and pressure. Lastly we observe that when suitably constructed apparatus is immersed in a field of sound waves, parts of the apparatus will yield to the momentum of the particles of the medium, or to the alternating excess pressures at a given point in the medium, with the result that the apparatus is driven into a state of vibration and so becomes a detector or meter of the sound energy which falls on it.

By a judicious application of mechanical theory, supported by experimental research, the science of Acoustics has been developed into a wide field of interesting phenomena with many useful applications. Some of these are purely mechanical, as for example, the use of high-frequency vibrating systems in submarine signalling; or again, the use of horns as an aid to sound radiation in loud-speaking apparatus. Some relate to physiology; mechanical theory has unquestionably been a valuable aid in the study of the mechanism of speech, and of hearing; and some applications are of psychological or

esthetic aspect, as they concern the power of the human ear to distinguish between ordered sounds such as music or speech and the disordered sounds which we call noise. At the present time it is possible to analyze and classify sounds of all kinds; and conversely, apparatus is available for generating and detecting sounds of almost any degree of complexity. By means of electrical transmitting apparatus sounds may be amplified or recorded with high precision. Sound, as a wave motion, is capable of interference and diffraction; these effects have been used to advantage to accomplish directive emission and detection. Even friction, or the dissipation of mechanical energy, has been turned to account to control resonance in vibrating systems, and the reverberations due to reflecting surfaces.

A firm foundation of mechanical theory is essential if we are to deal consistently with these varied phenomena. Lord Rayleigh's treatise, published thirty years ago, is still the authoritative statement [1] of the Principles; but there have been many contributions since (some due to Rayleigh himself), and the introduction of impedance methods, according to the modern practice, has been of the utmost advantage to the student of Acoustics. Hence the objective of the present text, which is to develop in its current form the indispensable minimum of established theory and show how effectively it lends itself to practical applications. The problems which we shall consider are as far as possible representative, but by no means do they cover the whole field of applied Acoustics. They have been selected from classical and more recent sources primarily for the purpose of illustrating, or providing a working substance for the theory.

The reader is supposed to have a fair acquaintance with the general principles of Physics, and but little need be said regarding the special facts which relate particularly to elementary experimental Acoustics; these matters are well treated else-

[1] The references to Rayleigh's "Theory of Sound" in the present volume are to the second edition (2 vols.), London, 1894 and 1896. Equally indispensable to the student, and a model of compactness in arrangement and style, is Prof. Lamb's "Dynamical Theory of Sound." Rayleigh's treatise has recently been reprinted, and a second edition of Lamb's "Sound" (London, 1925) has just appeared. The references to Lamb in what follows are to this work unless otherwise stated.

where.[1] The Analysis of Musical Sounds is doubtless familiar to the reader through the work of D. C. Miller.[2] An exhaustive collection of the data of Acoustics is soon to be available with the publication of the International Critical Tables; here will be included, for example the extended data on Speech and Hearing which have come from the staff of the Bell Telephone Laboratories.[3] The Critical Tables will also include the data of W. C. Sabine and later workers in the field of Architectural Acoustics.

In what follows we shall take for granted the usual assumptions of mechanics, but one principle, which is applied more frequently in Sound than in any other branch of Physics, deserves a special statement. This is the Principle of Superposition, according to which we may find the total displacement of a system acted on by a number of forces, by solving the problem for each applied force separately, and adding together all the resulting solutions. This follows from the fact that the differential equations of motion of such systems as we shall meet are linear; hence linear combinations of solutions of a given equation are solutions. Kinematically we shall sometimes have occasion to compound a number of vibrations of different periods to obtain the complete solution of a given problem. In some problems, the successive natural frequencies of the system are harmonically related, and to these cases the methods of Fourier's Series apply. But in other equally important problems, the component vibrations do not necessarily have frequencies which are simple multiples of one another; and the reader is warned that there is no limitation as to the relation between the component frequencies in applying the principle of superposition. The fact is that in the problems we shall encounter, there is always a solution in terms of a series of *normal functions*, each term of which represents a possible mode of vibration of the given system, and it is usually sufficient

[1] W. H. Bragg, "The World of Sound"; E. H. Barton, "Textbook of Sound"; Poynting and Thomson, "Textbook of Physics," Vol. II, "Sound." See also the Literaturverzeichniss in A. Kalähne's "Grundzüge der Akustik," Leipzig, 1910–1913.

[2] D. C. Miller, "The Science of Musical Sounds."

[3] See also the notes on Speech and Hearing in Appendix B.

to investigate only one typical term of the series, in order to understand the behavior of the system.[1]

2. *Equation of Motion of a Simple System*

We have observed that the first major problem in the theory of sound is that of producing sound waves by a "generator" or "vibrator" placed in contact with the sound-transmitting medium. To solve this problem in a general way we must determine the properties of the vibrating system in itself, the reactions of the medium upon it, and the properties of transmission in the medium. It is logical therefore to begin our studies with a treatment of the simplest type of vibrating system.

A particle of mass m is fastened to a stiff spring so that small motions take place in a horizontal line and the effect of gravity on the system can be neglected, also for the moment, any frictional "forces." Using the notation outlined at the beginning, the force required to produce a small displacement ξ from the equilibrium position is $s \cdot \xi$. The potential energy of the system for any displacement ξ is therefore

$$V = \tfrac{1}{2}s\xi^2,$$

and the kinetic energy of the system is

$$T = \tfrac{1}{2}m\dot{\xi}^2.$$

The total energy of the system at any instant is

$$W = T + V = \tfrac{1}{2}(m\dot{\xi}^2 + s\xi^2), \tag{1}$$

in which m and s are constant, and ξ, $\dot{\xi}$, functions of time. As there is no dissipation of energy due to friction or other causes, we can apply the "energy principle," whence we obtain

$$\frac{dW}{dt} = m\dot{\xi}\frac{d\dot{\xi}}{dt} + s\xi\frac{d\xi}{dt} = 0, \tag{2}$$

or in the convenient notation we have chosen, since $\dfrac{d\xi}{dt} \equiv \dot{\xi}$,

$$m\ddot{\xi} + s\xi = 0, \tag{2a}$$

[1] The subject of normal functions, of which the terms in a Fourier Series form a particular example, will recur in § 28.

which is the equation of motion of the system, under the action of no external forces.

If there is an external force $\Psi(t)$ acting on the system (that is on the mass m) its rate of doing work is clearly $\dot{\xi}\cdot\Psi(t)$ and to apply the energy principle in its general form we have, instead of equation (2)

$$\frac{dW}{dt} = \dot{\xi}\Psi(t) = m\dot{\xi}\cdot\ddot{\xi} + s\xi\cdot\dot{\xi}, \tag{2'}$$

whence the more general equation of motion

$$m\ddot{\xi} + s\xi = \Psi(t). \tag{2'a}$$

Consider now the effect of frictional forces, such as for example, the air friction on the mass m, as it oscillates back and forth. It is convenient, and often sufficiently accurate, to take the frictional force as directly proportional to the velocity of motion, i.e.:

$$\textit{frictional force} = r\cdot\dot{\xi},$$

in which r is the "resistance constant." Now, as the net effect of the frictional force $r\cdot\dot{\xi}$ is to oppose the action of the external force $\Psi(t)$ we have finally

$$m\ddot{\xi} + r\dot{\xi} + s\xi = \Psi(t), \tag{3}$$

for the complete equation of motion of the "system of one degree of freedom."

This equation specifies the behavior of the system under any applied force $\Psi(t)$, and with any initial conditions we choose to impose, so long as m, r and s remain constant. In practice this means that the displacement-coordinate ξ must not be forced to unduly large values; the "constants" of the system are then no longer constant and embarrassing discrepancies occur between the simple theory and the (more complicated) observed facts. In these cases it must be borne in mind that it is not equation (3) which is at fault: the simple theory is simply not applicable.

The analogy between equation (3) and the general equation

of the electric circuit containing inductance, resistance and capacity, namely

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = E(t), \qquad (i = \dot{q}) \tag{3'}$$

will not be labored here. It will doubtless interest the student who has mastered alternating-current theory to note all such analogies as we proceed; and we may commend Rayleigh, Vol. I, Chap. XB., and Maxwell, Vol. II, Chap. V–VII, as suitable collateral reading in this connection.

3. *Natural Oscillations*

We have to discover the important properties of the motion of the system by solving the linear differential equation (3). The simplest case is that in which an arbitrary displacement and velocity is given to the system which is then let go. In this case $\Psi(t) = 0$ and the solution of this equation contains two arbitrary constants, thus:

$$\xi = Ae^{\lambda_1 t} + Be^{\lambda_2 t}, \tag{4}$$

in which λ_1, λ_2, are the roots of the auxiliary equation. Forming the auxiliary equation from (3) and (4),

$$(\lambda^2 m + \lambda r + s)\xi = 0,$$

and solving,

$$\lambda = -\frac{r}{2m} \pm \sqrt{\frac{r^2}{4m^2} - \frac{s}{m}},$$

letting

$$\left.\begin{aligned} \frac{r}{2m} &= \Delta \quad \text{and} \quad \frac{s}{m} = n^2, \\ \text{we have:} \quad \lambda_1 &= -\Delta + i\sqrt{n^2 - \Delta^2} \equiv -\Delta + in', \\ \lambda_2 &= -\Delta - i\sqrt{n^2 - \Delta^2} \equiv -\Delta - in'; \end{aligned}\right\} \tag{5}$$

so that

$$\xi = e^{-\Delta t}(Ae^{in't} + Be^{-in't}). \tag{6}$$

This solution can be written

$$\begin{aligned}\xi &= e^{-\Delta t}[(A + B)\cos n't + i(A - B)\sin n't]\\ &= e^{-\Delta t}(A'\cos n't + B'\sin n't),\end{aligned}$$

or more concisely,

$$\xi = A_{1}e^{-\Delta t}\sin(n't + \theta), \tag{6'}$$

in which A_1 and θ are the arbitrary constants to be adjusted to the given boundary conditions. The reader may easily convince himself that all possible initial conditions can be fitted by (6′), and he may also derive the corresponding equation for the velocity:

$$\left.\begin{aligned}\dot{\xi} &= nA_{1}e^{-\Delta t}\cos(n't + \theta + \epsilon).\\ \epsilon &= \tan^{-1}\frac{\Delta}{n'} \equiv \tan^{-1}\frac{\Delta}{\sqrt{n^2 - \Delta^2}}.\end{aligned}\right\} \tag{6a}$$

Geometrically, (6′) is the equation of a sinusoidal curve whose amplitude diminishes with increasing time. The measure of this decay in amplitude is the constant Δ, called the "damping coefficient." The rate of decay depends on the ratio of resistance to mass, $\left(\frac{r}{2m}\right)$ but the rate of dissipation of energy $(r\dot{\xi}^2)$ in the system due to friction depends only on the resistance factor r. If there were no dissipation in the system, the frequency of the oscillations, known as free or natural oscillations in this case, would be $f_0 \equiv \frac{n}{2\pi}$; i.e., this is the frequency in which the system tends to oscillate, when suddenly excited. This frequency is diminished somewhat by the damping; the actual frequency of the oscillations is $f'_0 \equiv \frac{1}{2\pi}\sqrt{n^2 - \Delta^2}$. If Δ is made large enough, e.g., $\geqq n$ (by increasing r or diminishing m, and with it s), $\sqrt{n^2 - \Delta^2}$ may be made zero or imaginary, and free oscillations no longer are possible. To discuss this last case let $\xi = \xi_0$ and $\dot{\xi} = 0$, when $t = 0$; then applying (6) in the form

$$\xi = e^{-\Delta t}(Ae^{t\sqrt{\Delta^2 - n^2}} + Be^{-t\sqrt{\Delta^2 - n^2}}),$$

and calculating the constants A and B, we have

$$\xi = \frac{\xi_0}{2\sqrt{\Delta^2 - n^2}}[(\sqrt{\Delta^2 - n^2} + \Delta)e^{-(\Delta - \sqrt{\Delta^2 - n^2})t} + (\sqrt{\Delta^2 - n^2} - \Delta)e^{-(\Delta + \sqrt{\Delta^2 - n^2})t}]. \quad (7)$$

Thus the motion is the sum of two motions whose amplitudes decay at unequal, but usually rapid rates. It will be observed that this solution is not valid for exact aperiodicity (or *critical damping*), for then $\Delta = n$; the solution for this case is left to the reader as a part of problem 2, at the end of this chapter.

To summarize, we have found the natural or characteristic frequency of the system, that is, the frequency in which it tends to oscillate for an indefinite period of time; but we observe that for all practical purposes, the natural oscillations may be made to decay very rapidly, or to disappear altogether, if the damping factor is made very large.

4. *Periodic Driving Force*

Consider now the simple vibrating system under the action of a periodic force $\Psi(t) = \Psi_0 e^{\iota\omega t}$. The solution of the equation

$$m\ddot{\xi} + r\dot{\xi} + s\xi = \Psi_0 e^{\iota\omega t}, \quad (3)$$

will contain two parts, a "transient" and a "steady state" term. The former, obtained in the preceding section, expresses the temporary reaction of the system to any suddenly applied force, and in the long run its amplitude becomes very small. The steady state term expresses the tendency of the system to follow as well as it can, the driving force, and after the initial transient has disappeared the steady state term is a sufficiently complete description of the motion, unless further change in driving force is made. (At this stage we prefer to speak from the standpoint of physical experience, solving the problem step by step but it must be noted that in doing this we sacrifice a certain amount of rigor, for the sake of obtaining a more concrete picture.)

Neglecting any transient conditions, we may assume a

steady state motion $\xi = Ce^{i\omega t}$ to be a particular solution of the equation (3) above. Then, since $\dot{\xi} = i\omega\xi$ and $\ddot{\xi} = -\omega^2\xi$,

$$(-m\omega^2 + ir\omega + s)\xi = \Psi_0 e^{i\omega t},$$

and

$$\xi = \frac{\Psi_0}{i\omega Z} e^{i\omega t} \cdot \text{ in which } \quad Z = r + i\left(m\omega - \frac{s}{\omega}\right). \tag{8}$$

In the algebra of complex quantities (which is indispensable in dealing with problems such as this),

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{1}{\sqrt{a^2+b^2}} e^{-i\phi}, \quad \text{if} \quad \phi = \tan^{-1}\frac{b}{a}. \tag{9}$$

Hence the amplitude and velocity are respectively

$$\left.\begin{aligned} \xi &= \frac{\Psi_0}{i\omega Z} e^{i(\omega t - \phi)}, \\ &\text{and} \\ \dot{\xi} &= i\omega\xi = \frac{\Psi_0}{Z} e^{i(\omega t-\phi)}, \\ &\text{in which} \\ \phi = \tan^{-1}&\left(\frac{m\omega - \dfrac{s}{\omega}}{r}\right) \quad \text{and} \quad Z = \sqrt{r^2 + \left(m\omega - \frac{s}{\omega}\right)^2}. \end{aligned}\right\} \tag{10}$$

The reader need not be confused by the use of the symbol Z interchangeably for the complex impedance or its absolute value; it is always evident from the context which is meant.

Before going further note:

(*a*) The 90° difference in phase between ξ and $\dot{\xi}$;

(*b*) The phase lag ϕ between driving force and resultant velocity;

(*c*) The quantity Z, known as the impedance, which has a minimum value $Z_0 = r$ when

$$\omega^2 = \frac{s}{m} \,[= (2\pi f_0)^2, \ \S\, 3].$$

We shall return to a more detailed discussion of these later.

5. *Complete Solution for Forced and Free Oscillations*

To find the most general behavior of the system we must add together the solutions for free and forced vibrations and make them fit such boundary conditions as we desire to impose. Usually the forced vibrations are the more important in practical calculations so consider these first. In practice the driving force is usually specified by a series whose typical term is

$$C_1 \cos \omega_1 t \quad \text{or} \quad D_1 \sin \omega_1 t,$$

so we sacrifice little by using (say) only the real component of the rotating unit vector $e^{i\omega t}$, e.g., letting $\Psi = \Psi_0 \cos \omega t$ for the forced vibrations, whence:

$$\left.\begin{aligned} \dot{\xi} &= \frac{\Psi_0}{Z} \cos (\omega t - \phi), \\ \text{and} \qquad \xi &= \frac{\Psi_0}{\omega Z} \sin (\omega t - \phi). \end{aligned}\right\} \tag{10a}$$

It is clear that the imaginary components, used consistently all the way, would have equally sufficed, the result being merely an interchange of sine for cosine, with due regard to signs.

Adding the transient terms (equations 6′, 6*a*, §3), we have:

$$\left.\begin{aligned} \xi &= Ae^{-\Delta t} \sin (n't + \theta) + \frac{\Psi_0}{\omega Z} \sin (\omega t - \phi), \\ \dot{\xi} &= nAe^{-\Delta t} \cos (n't + \theta + \epsilon) + \frac{\Psi_0}{Z} \cos (\omega t - \phi). \end{aligned}\right\} \tag{11}$$

Equations (11) with due adjustment of A and θ to fit boundary conditions, describe the motion of the system under the excitation of a simply periodic force. Applying the methods of the Fourier Series and Integrals a sum of such solutions is always possible, which will fit any boundary conditions and a driving force of any nature whatsoever.

6. *Initial Conditions Under Periodic Driving Force*

A good enough understanding of the behavior of the system under periodic excitation can be had if we study equations (11) first as functions of time, and second as functions of the frequency with which the system is driven.

Considering frequency $\left(f = \dfrac{\omega}{2\pi}\right)$ constant, let the system start from rest, i.e.,

$$\xi = \dot{\xi} = 0 \quad \text{when} \quad t = 0.$$

For these conditions,

$$\left.\begin{aligned} 0 &= A_1 \sin\theta + \frac{\Psi_0}{\omega Z}\sin(-\phi) && \text{(amplitude),} \\ 0 &= nA_1 \cos(\theta + \epsilon) + \frac{\Psi_0}{Z}\cos(-\phi) && \text{(velocity).} \end{aligned}\right\} \qquad (12)$$

The rigorous solution of these equations for A and θ is much more cumbersome than it appears offhand, and to make progress with the matter approximations must be made. ϵ is usually small $\left(\sin\epsilon = \dfrac{\Delta}{n}, \S\ 3\right)$ and may be sacrificed in the interest of symmetry. We then have, squaring and adding

$$A_1^2(\sin^2\theta + \cos^2\theta) = \frac{\Psi_0^{\,2}}{\omega^2 Z^2}\left(\sin^2\phi + \frac{\omega^2}{n^2}\cos^2\phi\right) = \frac{\Psi_0^{\,2}}{\omega^2 Z^2}\cdot Q^2.$$

Choosing the negative root, as applicable to the conditions,

$$\left.\begin{aligned} &A_1 = -\frac{\Psi_0}{\omega Z}\cdot Q, \qquad Q = \sqrt{\sin^2\phi + \frac{\omega^2}{n^2}\cos^2\phi}; \\ &\sin\theta = -\frac{1}{Q}\sin\phi, \ \cos\theta = \frac{\omega}{n}\frac{1}{Q}\cos\phi. \end{aligned}\right\} \qquad (13)$$

Thus for $\omega = n$ (i.e., $f = f_0$) θ is the negative of ϕ, $Q = 1$ and the amplitude A_1, at $t = 0$, is the negative of the steady state amplitude $\frac{\Psi_0}{\omega Z}$ of the forced vibration which is finally attained.

But in general,

$$\left.\begin{aligned} \xi &= \frac{\Psi_0}{\omega Z}\Big[\sin(\omega t - \phi) - Qe^{-\Delta t}\sin(n't + \theta)\Big], \\ \dot{\xi} &= \frac{\Psi_0}{Z}\Big[\cos(\omega t - \phi) - \frac{n}{\omega}Qe^{-\Delta t}\cos(n't + \theta + \epsilon)\Big]; \end{aligned}\right\} \quad (14)$$

the indicated solution for θ from (13) being understood. The history of the system from $t = 0$ is as follows: starting from rest, the amplitude (or velocity, if you prefer) can be considered first as the sum of a "transient" amplitude and a "steady state" amplitude equal in numerical value but opposite in sign. They are exactly opposed, strictly speaking, only at the start; for in general $n \neq n' \neq \omega$. During the early stages of the motion, the transient vibration, steadily weakening in amplitude, is beating with the forced vibration, and the observable result of this interference is a gradual asypmtotic rise in the total amplitude, reaching its steady state value when the transient has disappeared. During this rise in amplitude the (apparent) frequency of the vibration is neither that of the transient nor that of the impressed force; it is a variable composite of the two which approaches in value the steady state frequency as the transient disappears. In listening, for example to a receiver diaphragm, the ear can often distinguish the transient tone when a periodic driving force is suddenly applied, particularly if the driving force has a frequency which differs considerably from the principal natural frequency of the diaphragm.

7. *Periodic Driving Force with Variable Frequency*

Consider the steady state of vibration established and let the frequency of the driving force be varied. The velocity and

amplitude (equations (10), (11)) are:

$$\left.\begin{aligned} \dot{\xi} &= \frac{\Psi_0}{Z}\cos(\omega t - \phi), \\ \xi &= \frac{\Psi_0}{\omega Z}\sin(\omega t - \phi); \end{aligned}\right\} \quad (15)$$

in which

$$Z = \sqrt{r^2 + \left(m\omega - \frac{s}{\omega}\right)^2}, \quad \phi = \tan^{-1}\frac{m\omega - \dfrac{s}{\omega}}{r}.$$

Or, focussing the attention on maximum values only

$$\dot{\xi}_0 = \frac{\Psi_0}{\sqrt{r^2 + \left(m\omega - \dfrac{s}{\omega}\right)^2}}, \qquad \xi_0 = \frac{\dot{\xi}_0}{\omega}. \quad (16)$$

For constant Ψ_0, the following table gives values for the various factors from which the reader may infer the way in which the behavior of the system depends on the driving frequency.

f	ω	$\lvert Z\rvert$	$\left\lvert\frac{1}{Z}\right\rvert$	$\left\lvert\frac{1}{\omega Z}\right\rvert$	ϕ
0	0	∞	0	$\frac{1}{s}$	$-\frac{\pi}{2}$
$\frac{1}{2\pi}(n-\Delta)$	$(n-\Delta)$*	$\sqrt{2}r$	$\frac{1}{\sqrt{2}r}$	$\frac{1}{\sqrt{2}(n-\Delta)r}$	$-\frac{\pi}{4}$
$f_0=\frac{n}{2\pi}$	$n=\sqrt{\frac{s}{m}}$	r	$\frac{1}{r}$	$\frac{1}{nr}$	0
$\frac{1}{2\pi}(n+\Delta)$	$(n+\Delta)$*	$\sqrt{2}r$	$\frac{1}{\sqrt{2}r}$	$\frac{1}{\sqrt{2}(n+\Delta)r}$	$+\frac{\pi}{4}$
∞	∞	∞	0	0	$+\frac{\pi}{2}$

* Approximately; valid when Δ^2 is small as compared to n^2.

The reader may check these values, and plot ξ_0 and $\dot{\xi}_0$ against frequency. If the frequency scale is logarithmic, or by octaves, the curve for the *velocity* will be symmetrical about the point

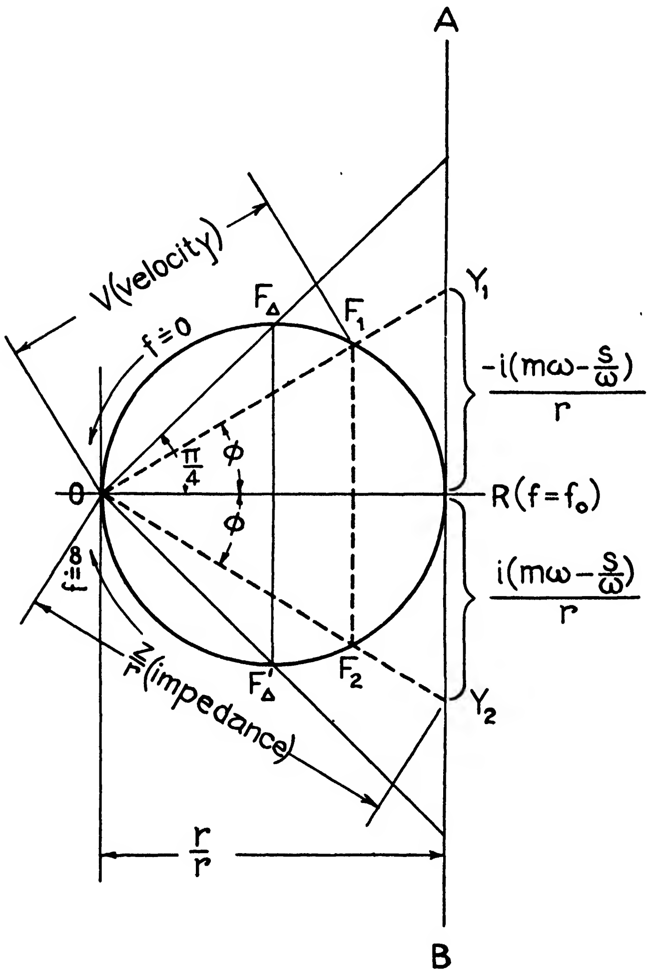


FIG. 1.—CIRCLE DIAGRAM OF IMPEDANCE.

$f = f_0$. This is the frequency of maximum velocity, or in the vernacular the "resonance frequency." If the vertical scale for the amplitude is made n-times as great as that for the velocity,

the maxima of the two curves at $f = f_0$ will nearly coincide; but for $0 \leq f \leq f_0$ the curve for *amplitude* will lie *above* the velocity curve, and conversely for points above resonance, it will lie *below* the velocity curve. It is not symmetrical with respect to f_0; and the amplitude at $f = 0$ (compare the velocity) is $\frac{\Psi_0}{s}$, as is evident from "static" or "equilibrium" theory.

The difference $\frac{2\Delta}{2\pi}$ between the frequencies for which $\phi = \pm \frac{\pi}{4}$ is the commonly accepted measure of the sharpness or bluntness of the resonance curve. For responding well to a wide variation in frequency in the driving force, bluntness of resonance or tuning in the system is desirable; this is obtained by making the damping large, with the collateral advantage of minimizing the interference due to natural vibrations or transients—hence the utility of such systems. Sharply tuned systems have a definite place in single-frequency apparatus, such as for example tuning forks and resonators generally.

Experimentally, the damping of a bluntly tuned system is best obtained from the shape of the resonance curve of velocity; for the sharply tuned system, the damping should be determined from the rate of decay of the natural oscillations.

The phase relations between force, impedance and velocity are clearly shown in the circle diagram [1] first used by Kennelly

[1] *Proc. Am. Acad. Arts and Sci.*, 48, 1912, p. 113. When the telephone receiver is driven by alternating current a counter alternating e.m.f. is generated by the oscillations of the diaphragm. This counter e.m.f. is proportional to the velocity of the diaphragm, and hence to the driving current, at any given frequency. It manifests itself as a component of the impedance of the machine, which component is the difference between the impedance when the moving member is free to oscillate, and the impedance when motion is prevented. Kennelly and Pierce applied the term "motional impedance" to this component and showed how it could be experimentally determined; they also showed how, by a transformation of dimensions and phase, the motional impedance measures the velocity of the vibrating member, as a function of frequency.

The circle diagram is in wide use in the current literature of electrical vibrating apparatus, many references to which are given in Appendix B. For example, see Kennelly and Taylor, *Proc. Am. Phil. Soc.*, 55, 1916, p. 415; Hahnemann and Hecht, *Phys. Zeit.*, 20, 1919, p. 104 and *ibid.*, 21, 1920, p. 264; also R. L. Wegel, *J.A.I.E.E.*, 1921, p. 791.

and Pierce, in their study of the telephone receiver. In its simplest form this is shown in Fig. 1. In vector notation:

$$Z = r + i\left(m\omega - \frac{s}{\omega}\right),$$

i.e., if

$$Z = |Z|\, e^{i\phi},$$

$$\frac{1}{Z} = \frac{1}{|Z|}\, e^{-i\phi};$$

and to simplify matters, let us study for a steady alternating force, the ratio of velocity at frequency f, to velocity at resonance (f_0), i.e.,

$$V = V_0 e^{-i\phi} = \frac{r}{r + i\left(m\omega - \frac{s}{\omega}\right)} = \frac{r}{Z}. \tag{17}$$

A distance $OR = \frac{r}{r}$ = unity is laid off on the x-axis and this is taken as the velocity vector *at resonance*, measured in *arbitrary units*. On this as a diameter, construct the circle, and draw the tangent BRA parallel to the y axis. Then for any line OF_1Y_1, $\bar{O}F_1$ is the reciprocal of $O\bar{Y}_1$, a familiar property of inversion between points on the line and points on the tangent circle, of unit diameter. For some frequency $f < f_0$ lay off the vector RY_2 equal to the ratio of reactance to resistance. Then $OY_2 = \frac{Z}{r}$ and OF_1 (the mirror image of OF_2) is equal to $\frac{r}{Z}$. Thus plotting the reactances for all values $0 \leq f \leq \infty$ the end-point F_1 of the velocity vector will travel around the circumference of the circle, passing through the point R when $f = f_0$. The points F_Δ and F'_Δ will be recognized as those corresponding to the frequencies $f_0 - \frac{\Delta}{2\pi}$ and $f_0 + \frac{\Delta}{2\pi}$, respectively.

A somewhat similar diagram could be plotted for the amplitude vector, but it must be noted that the locus of its end point is *not* a circle. This curve, which is nearly closed, must be turned about O through an angle $-\frac{\pi}{2}$ to allow for the lag of the amplitude with respect to the velocity. For further diagrams of amplitude and velocity vectors the reader may refer to Kennelly's "Electrical Vibration Instruments" (Macmillan, 1923).

The circle diagram sums up the kinematics of the steady state theory of the simple vibrating system.

8. *Physical Nature of the Constants of the System*

Turn now from kinematical considerations (in which m, r, and s are mere parameters in an equation) to a more physical view of the nature of these "constants." Practically, each of these factors is an "effective" or mean value, which to a certain degree of approximation replaces the total effect of inertia or resistance or stiffness of the system, each factor being summed up or averaged with respect to the coordinate ξ in terms of which the motion of the system is to be described. Thus a variety of geometrical and mechanical considerations enter into the determination of each.

Much progress has been made in the study of the telephone diaphragm, for example, by assuming that such an essentially complicated system can be replaced by an equivalent simple system with constant m, r, and s. Logically this assumption has no foundation whatsoever, because in order to determine m and s a priori the form which the diaphragm takes in bending must be determined, and this, particularly when the stress is not uniformly applied over the surface of the diaphragm is in itself a mechanical problem of great difficulty. The diaphragm is really a system of a large number of degrees of freedom and consequently its form of vibration will vary with frequency; there is no such thing as a constant mass factor for a diaphragm. The full significance of these statements will appear more

clearly when we come to consider systems of several degrees of freedom; it is sufficient for our present purpose to point out the limitations of the simple theory. The methods sketched in this chapter for dealing with actual systems often suffice to give a fairly clear picture of the behavior of the system, and this is their sole justification.

In the case of resistance, the matter is even more complicated. The resistance, or friction coefficient in the case of bending a rod or plate must include not only the internal friction, due to the sliding of all the elements of the structure with respect to each other, but air-resistance factors as well, or in more general terms the effect of immersing the vibrating system in the medium. In the first place sound energy is radiated from the system, whenever the system is in contact with the medium, and this represents a steady drain on the energy supplied to the system. The rate at which the system works on the medium by virtue of radiation is, as we shall see later, $R\dot{\xi}^2$ in which R has the dimensions of resistance. R is an important constant characteristic of the medium; it may be called the radiation resistance.

In addition to radiation there is usually a pure-friction term due to lamellar motion in the surrounding air, in which the viscosity of the medium comes into play. The resistance in the case of the simple system we have studied was due entirely to this effect, and it is possible, in case great damping is required so to arrange the system that air friction or viscosity is utilized especially for this purpose. If we consider the frictional air resistance to be included in the inherent resistance of the system, then the total resistance against which the system works is

$$r = r_{\text{internal}} + R_{\text{radiation}},$$

the rate of expenditure of energy being

$$r\dot{\xi}^2 = (r_1 + R)\dot{\xi}^2. \tag{18}$$

The electrical analogy is of course evident.

It is an old contention that the opposing force due to friction may not be strictly proportional to the velocity of motion. Experimentally we believe that the assumption of the linear relation is fairly well substantiated, for the small velocities we deal with in sound and vibrating systems. It is true that for rapidly moving bodies in air (e.g., artillery projectiles) there are large amounts of energy dissipated in eddies (i.e., in kinetic energy effectually abstracted from the energy of the projectile) and the corresponding dissipative reaction or force in this case is proportional to the square of the velocity. But as long as we are dealing with pure stream-line motion and small velocities we are correct in writing

$$\textit{Frictional force} = \textit{Const.}\cdot\mu\cdot\dot{\xi}. \quad (\mu = \text{viscosity coefficient.})$$

A good example of viscosity-damping will be given in a succeeding article.[1]

Finally, to complete our discussion of the nature of resistance and stiffness, we note the effects of forcing the system to oscillate at such large amplitudes that elastic hysteresis is brought into play. In this case both r and s depend on the amplitude of the motion, and the simple theory is of no avail; power series $r = f_1(\xi)$ and $s = f_2(\xi)$ must be substituted for the simple constants and the solution of the problem is so difficult that special methods are required. The treatment of such cases is beyond the scope of the present outline; the reader may refer to Rayleigh (§§67, 68) for suggestions as to theory, for the simple system. In Rayleigh, Appendix to Chapter V (II, p. 480), some of the properties of non-linear vibrations in compound systems are investigated.[2]

[1] The reader interested in viscosity, hydrodynamical resistance, etc., may consult Chaps. III, IV, V of "The Mechanical Properties of Fluids," Drysdale et als., Van Nostrand, 1924. This is a very useful book and will be referred to again.

[2] Following Rayleigh (§§67, 68), the reader may refer to Part II of a paper by E. Waetzmann, *Phys. Zeit.*, XXVI, 1925, p. 740, on "Modern Problems of Acoustics." This part of the paper deals with non-linear oscillations, difference tones, etc., and gives references to Waetzmann's earlier work.

9. *Equilibrium or Low-Frequency Theory; Circular Membrane*

The static or equilibrium theory (more logically speaking, the low-frequency theory) is often of considerable utility in studying such problems as the vibration of diaphragms and the like. We shall illustrate its application to an interesting problem, and incidentally gain some notion of the determination of mass and stiffness constants.

A thin circular membrane, with fixed boundary, may vibrate in a number of modes. Of those that are symmetrical with respect to the axis we shall investigate the fundamental or gravest mode. Obviously if we can gain a fair idea of the shape of the membrane in its position of maximum distension we can make approximate computations of mass and stiffness factors, and so determine the natural frequency of this mode of vibration.

For a circular membrane of no inherent stiffness and loaded with a uniform force P per unit area, it is trivial to show that, for small displacements,

$$\xi = \frac{Pa^2}{4\tau}\left(1 - \frac{r^2}{a^2}\right) = \xi_0\left(1 - \frac{r^2}{a^2}\right), \tag{19}$$

in which a is the radius of the fixed boundary, τ the tension, and ξ_0 the central or maximum displacement. Let us assume that this paraboloidal form of displacement is substantially maintained up to the frequency of the gravest mode; and determine m and s in terms of $\ddot{\xi}_0$ and ξ_0. Summing up the annular elements of mass $2\pi\rho r dr$ we have for the kinetic energy of the system

$$T = \tfrac{1}{2}m\dot{\xi}_0^2 = \tfrac{1}{2}\int_0^a \dot{\xi}_0^2\left(1 - \frac{r^2}{a^2}\right)^2 2\pi\rho r dr = \tfrac{1}{2}(\tfrac{1}{3}\pi\rho a^2)\dot{\xi}_0^2. \tag{20}$$

Thus the mass constant is one third of the total mass of the diaphragm. The potential energy can be calculated from the work done by the total force in distending the diaphragm, taking care to allow for the variation of P with ξ_0. Since

$$P = \frac{4\tau\xi_0}{a^2} \quad \text{and} \quad d\xi = \left(1 - \frac{r^2}{a^2}\right)d\xi_0,$$

$$V = \int_0^{\xi_0}\int_0^a 2\pi r dr P d\xi = \frac{8\pi\tau}{a^2}\int_0^a\left(1 - \frac{r^2}{a^2}\right)r dr\int_0^{\xi_0}\xi_0 d\xi_0,$$

that is,

$$V = \frac{8\pi\tau}{a^2}\left(\frac{a^2}{4}\right)\frac{\xi_0^{\,2}}{2} = \pi\tau\xi_0^{\,2} = \tfrac{1}{2}s\xi_0^{\,2} \quad \text{whence} \quad s = 2\pi\tau. \qquad (21)$$

The natural frequency is then (neglecting damping),

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{6\tau}{\rho a^2}} = \frac{1.22}{\pi a}\sqrt{\frac{\tau}{\rho}}. \qquad (22)$$

This is a surprisingly accurate result as will appear by comparison with that obtained more rigorously in the next section.

10. *General Theory of the Circular Membrane; Bessel's Functions*

This problem of the symmetrical vibration of the circular membrane is of some importance and its general solution involves mathematical devices which are among the most interesting and useful in analytical mechanics. Neglecting damping as before, we adapt to our notation (§ 9) the straightforward treatment of Lamb (Sound, § 54):

The stress across a circle of radius r has a resultant $\tau\cdot 2\pi r\cdot\frac{\partial\xi}{\partial r}$ normal to the plane of the undisturbed membrane and the difference between the stresses on the edges of the annulus whose inner and outer radii are r and $r + dr$ gives a force

$$\frac{\partial}{\partial r}\left(\tau\cdot 2\pi r\cdot\frac{\partial\xi}{\partial r}\right)dr.$$

Equating this to $\rho\cdot 2\pi r dr\cdot\ddot{\xi}$ which is the rate of change of momentum of the annulus, we have

$$\rho\ddot{\xi} = \frac{\tau}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\xi}{\partial r}\right). \qquad (23)$$

Now if we assume for the solution of (23)

$$\xi = A\Phi(r)\cdot e^{int},$$

the factor depending on t can be removed from the differential equation, and we obtain

$$\left(\frac{\partial}{\partial^2 r} + \frac{1}{r}\frac{\partial}{\partial r} + k^2\right)\Phi(r) = 0, \tag{24}$$

in which $k^2 = \frac{n^2\rho}{T}\cdot$ This is Bessel's equation (the simplest form) and if we try solutions of the type

$$\Phi(kr) = a_0 + a_1 kr + a_2 k^2 r^2 + \ldots$$

and determine the coefficients by substitution in (24) we find

$$\Phi(kr) = J_0(kr) \equiv 1 - \frac{k^2r^2}{2^2} + \frac{k^4r^4}{2^2 4^2} - \ldots, \tag{25}$$

which function is known as the Bessel's function of the first kind, of zero order. The general solution of (24) has two arbitrary constants and is written

$$\xi = [AJ_0(kr) + BK_0(kr)]e^{int}, \tag{26}$$

in which K_0 is one of the Bessel's functions of the second kind of zero order. $K_0(kr)$ becomes infinite for $r = 0$ and obviously does not satisfy the boundary conditions, therefore $B = 0$. The solution of the problem (since $\xi = \xi_0 e^{int}$ for $r = 0$ and $\xi = 0$ for $r = a$) depends on those values of k which satisfy the equation

$$\xi_a = \xi_0 J_0(ka) = 0, \quad \text{i.e.,} \quad J_0(ka) = 0. \tag{27}$$

By reference to tables (see Gray and Mathews, "Bessel Functions," or Rayleigh, Vol. I, p. 321) we find, for the roots of (27)

$$\frac{ka}{\pi} = .766,\ 1.757,\ 2.755.\ \ldots$$

Thus using the value $k_1 = .766\frac{\pi}{a}$ for example and letting $0 \leq r \leq a$

we can determine the shape of the distended membrane for the lowest natural frequency from the equation

$$\xi = \xi_0 J_0(k_1 r) = \xi_0 J_0\left(.766\pi\frac{r}{a}\right), \tag{27a}$$

and the natural frequency in this mode of vibration is

$$f_1 = \frac{n_1}{2\pi} = \frac{k_1}{2\pi}\sqrt{\frac{\tau}{\rho}} = \frac{1.20}{\pi a}\sqrt{\frac{\tau}{\rho}}. \tag{28}$$

This result is in close agreement with that obtained previously (eq. 22, § 9), assuming (what is substantially correct) a para-

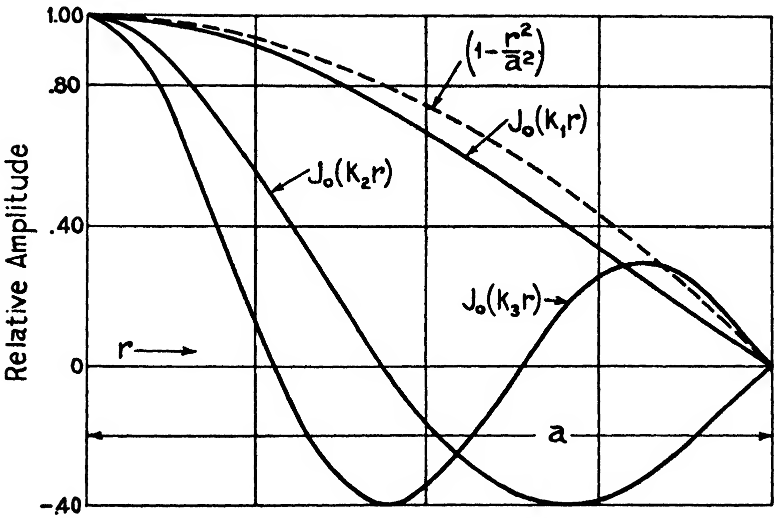


FIG. 2.—FORM OF VIBRATING CIRCULAR MEMBRANE IN TERMS OF BESSEL'S FUNCTIONS.

boloidal form for the distended membrane in the fundamental mode of vibration.

In order to get a better idea of the behavior of the membrane, there have been plotted, in Fig. 2, the three functions

$$\frac{\xi}{\xi_0} = J_0(kr), \qquad k = k_1,\ k_2,\ k_3;$$

for values ($0 \leq r \leq a$). A dotted line indicates the section of the paraboloid which is nearly equivalent to $J_0(k_1 r)$. For further

graphs of the Bessel's functions reference may be had to the "Funktionentafeln" of Jahnke and Emde.

The discussion now takes a more general aspect, to account for the introduction of damping into the system. Let a resistance constant R per unit area account for dissipation by friction and radiation; then instead of (23) we have

$$\frac{R\dot{\xi} + \rho\ddot{\xi}}{\tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \xi}{\partial r}\right). \tag{23a}$$

Assuming

$$\xi = A\Phi(r)e^{\lambda t},$$

we have

$$\left.\begin{aligned} &\left(\frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k^2\right)\Phi(r) = 0, \\ &\rho\lambda^2 + \lambda R + k^2\tau = 0. \end{aligned}\right\} \tag{24a}$$

in which

Thus (cf. § 3),

$$\left.\begin{aligned} \xi &= J_0(kr)(Ae^{\lambda_1 t} + Be^{\lambda_2 t}), \\ \lambda_1 &= -\Delta + in', \quad \lambda_2 = -\Delta - in'; \\ n' &= \sqrt{n^2 - \Delta^2}, \quad n^2 = k^2\frac{\tau}{\rho}. \end{aligned}\right\} \tag{26a}$$

in which

Now suppose the membrane given an initial distortion of the form $\xi_0 J(k_j r)$, and let go; then it is clear that with the proper adjustment of A and B (cf. 6′) we have for the amplitude

$$\xi = \xi_0 J_0(k_j r)e^{-\Delta t}\cos n'_j t, \tag{29}$$

in which $(k_j r)$ is a root of the J_0 as before. The natural frequencies (as modified by damping) are now determined by

$$f_1 = \frac{1}{2\pi}n'_1 = \frac{1}{2\pi}\sqrt{\frac{k_1{}^2\tau}{\rho} - \Delta^2}, \quad \text{etc.} \tag{28a}$$

If, instead of being distended into a shape corresponding to one of its normal modes the membrane is initially given an

arbitrary distension (as for example the form of a cone) it is possible (as can be proved in a way analogous to the proof of the Fourier synthesis) to build up a series

$$\xi = A_1 J_0(k_1 r) e^{-\Delta_1 t} \cos (n_1 t + \theta_1) + A_2 J_0(k_2 r) e^{-\Delta_2 t} \cos (n_2 t + \theta_2) + \ldots \quad (29)$$

which will satisfy all the conditions of the problem. The vibration in this case, until it is completely extinguished by damping, is thus a composite of all the possible symmetrical natural vibrations of the system, the relative amplitudes A_1, A_2, and phases θ_1, θ_2, . . . being chosen so as to build up, by superposition, the arbitrary initial shape of the membrane.

Suppose now that the membrane is driven by a force per unit area $\Psi_0 e^{i\omega t}$; we wish to determine the general properties of the motion of the membrane as the driving frequency is varied. Instead of equation (23*a*), we have

$$R\dot{\xi} + \rho\ddot{\xi} = \frac{\tau}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \xi}{\partial r}\right) + \Psi_0 e^{i\omega t}, \quad (30)$$

and if $\dot{\xi} = i\omega\xi$ and $\ddot{\xi} = -\omega^2\xi$ (steady state theory),

$$\left.\begin{aligned} &\frac{\partial^2 \xi}{\partial r^2} + \frac{1}{r}\frac{\partial \xi}{\partial r} + \kappa^2 \xi = -\frac{\Psi_0}{\tau} e^{i\omega t}, \\ \text{in which} \quad &\kappa^2 = \frac{\rho\omega^2 - i\omega R}{\tau}. \end{aligned}\right\} \quad (31)$$

The right-hand member of the equation (31) is not a function of the radius r; and if the right-hand member of the equation were zero, then $AJ_0(\kappa r)$ would be a solution of the equation. We therefore conclude that the solution of (31) is

$$\xi = -\frac{\Psi_0 e^{i\omega t}}{\kappa^2 \tau} + AJ_0(\kappa r). \quad (32)$$

Determining A for the boundary condition $\xi(a) = 0$,

$$A = \frac{1}{J_0(\kappa a)} \frac{\Psi_0 e^{i\omega t}}{\kappa^2 T};$$

the amplitude is

$$\xi(r) = \left[\frac{J_0(\kappa r)}{J_0(\kappa a)} - 1\right]\frac{\Psi_0 e^{i\omega t}}{\kappa^2 T}, \tag{33}$$

and the velocity is (cf. eq. (10)):

$$\dot{\xi}(r) = -\left[1 - \frac{J_0(\kappa r)}{J_0(\kappa a)}\right]\frac{i\omega\Psi_0 e^{i\omega t}}{\kappa^2 T}. \tag{33a}$$

In this expression for the velocity, the impedance per unit area is

$$Z(r) = \frac{\kappa^2 T}{-i\omega\left[1 - \dfrac{J_0(\kappa r)}{J_0(\kappa a)}\right]} = \frac{R - i\rho\omega}{\left[1 - \dfrac{J_0(\kappa r)}{J_0(\kappa a)}\right]}. \tag{33b}$$

To fully discuss equations (33–33b) and so determine the behavior of the membrane when the driving frequency is varied is a difficult matter because of the complex variable κr which is the argument of the Bessel's function. It can be seen at once that for zero frequency (since $\kappa_0 = 0$ and $J_0(0) = 1$) $Z = \infty$ and the velocity is zero. If it were not for the resistance factor R which we have introduced into κ^2, we should have $J_0(\kappa a) = 0$ whenever $\omega = n_1, n_2$, etc. ($n_1, n_2 \ldots$ each being 2π times a resonant frequency of the membrane) and for these frequencies Z would vanish, giving an infinite velocity. Without rationalizing Z and computing its absolute value, it is evident that Z tends to show a series of *minimum* values corresponding to these points, and these minimum values rise as R increases. It will also be noted that as the mode of vibration changes with frequency, mass and stiffness constants must vary with frequency. The curve of Fig. 3, showing the variation of the mass constant is due to R. L. Wegel, and was made in connection with his study of the telephone receiver.

Equation (33*b*) was obtained to fit the boundary condition $Z = \infty$ when $r = a$. An interesting test of (33*b*) can be made by computing for low frequencies the impedance per unit area at the *center of the diaphragm*, for which $J_0(\kappa r) = 1$. Since we know that the membrane takes a paraboloidal form under uniform

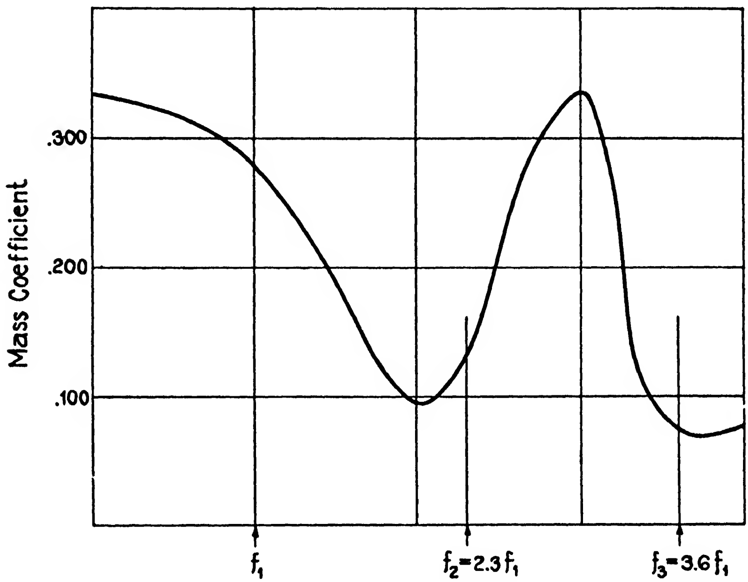


FIG. 3.—MASS COEFFICIENT IN TERMS OF TOTAL MASS FOR CIRCULAR MEMBRANE.

pressure at low frequencies, and departs only slightly therefrom even at the first resonant frequency, we need use only two terms of the Bessel's function, i.e.,:

$$J_0(\kappa a) \doteq 1 - \frac{\kappa^2 a^2}{4}, \qquad \text{cf. (25)}$$

and we have

$$Z_{r=0} = \frac{\kappa^2 \tau}{-i\omega\left[1 - \dfrac{1}{1 - \dfrac{\kappa^2 a^2}{4}}\right]}$$

$$= +\frac{1}{i\omega}\left[\frac{4\tau}{a^2} - \tau\kappa^2\right] = R + i\left[\rho\omega - \frac{4\tau}{a^2\omega}\right]. \qquad (34)$$

Thus the impedance appears in the standard form, and we recognize at once a stiffness constant, $s_{r=0} = \frac{4\tau}{a^2}$, a result consistent with equation (19). This is to say that if a small portion at the center of the membrane could be made to vibrate, piston-like, with the mass and stiffness constants that appear in (34) its natural frequency would be $\frac{1}{\pi a}\sqrt{\frac{\tau}{\rho}}$, which is not quite the natural frequency of the membrane as a whole, in its fundamental mode. The impedance per unit area is a minimum at the center, rising to ∞ at $r = a$ and the stiffness constant per unit area increases while the mass constant per unit area decreases as r increases from zero to the value a.

The complete discussion of all the modes of vibration of the circular membrane is readily available in the classical literature. Together with the theory of the circular plate, it is of fundamental importance in the specialized study of the telephone receiver diaphragm.

For the theory of Bessel's Functions, the reader may consult Chap. XVII of "Modern Analysis," by E. T. Whittaker and G. N. Watson (Cambridge, 1920), in addition to references previously given.

11. *Air Damping; Piston System; Ber and Bei Functions*

In the case of the symmetrical vibration of a membrane it was noted that a small portion at the center could be considered as a piston which, in its motion to and fro, was always perpendicular to the axis of symmetry. In many problems it is convenient to replace the whole active portion of a diaphragm by an imaginary "equivalent piston," this device being legitimate only for those frequencies for which all parts of the diaphragm move strictly in phase with one another. Thus in the low frequency theory we virtually replaced the membrane (§ 9) by a piston whose mass was one-third the total mass of the membrane, and whose total stiffness constant ($2\pi\tau$) was one-half the stiffness constant at the center of the

membrane $\left(\text{i.e., } \frac{4T}{a^2}\right)$ multiplied by the area of the diaphragm. Thus a piston of a certain area, having the mass and stiffness constants as given, and oscillating according to the equation $\xi = \xi_0 \cos nt$ well represents the motion of the membrane, at frequencies below the first natural frequency. At higher frequencies this substitution is no longer valid. The basis on which the area of the equivalent piston should be determined is set forth in the next section.

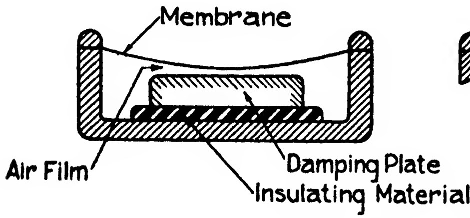


FIG. 4.—SECTION OF CONDENSER TRANSMITTER.

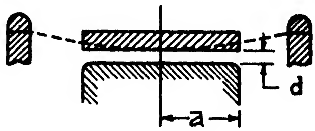


FIG. 4a.—EQUIVALENT PISTON SYSTEM.

Let us now consider a problem in variable stiffness and resistance over the surface of a membrane due to certain constraints: the investigation being simplified by the use of an equivalent piston whose area is arbitrarily fixed.

The structure of a simple form of condenser transmitter is shown in section in Fig. 4. The diaphragm is a thin stretched membrane whose stiffness, already great (due to the tension), is further increased by the compression of the air between the membrane and the "damping plate" below it, with the result that the first resonant frequency is very high. Thus there is a wide working range of frequency in which the diaphragm vibrates very nearly in its first normal mode. Due to the static charge on the system as a condenser, the membrane is attracted by the damping plate and takes in this equilibrium position a nearly paraboloidal form; such small oscillations as are produced in it, by the impact of sound waves, are virtually increases or decreases of this paraboloidal form of distension, as long as the frequency is within the working range. It seems

reasonable, therefore, to replace the central or active portion of the membrane by a piston (as in Fig. 4*a*), and as the curvature of the membrane is small in this region, the piston can be taken as a plane disc which, in its mean position, is separated by a certain distance d from the damping plate. Motion of the piston produces in the thin air film a radial displacement of the air, which is impeded by its viscosity. At low frequencies, the air having ample time to escape, there is a maximum amount of dissipative or frictional reaction, and a minimum of elasticity or stiffness due to accumulated pressure on the film. At high frequencies this situation is reversed, little air can escape, and the stiffness reaction increases rapidly to a maximum value.

The piston is given a small displacement $\xi = \xi_0 \cos \omega t$. Assuming that the temperature of the air remains constant (i.e., that the air remains in thermal equilibrium with its metallic bounding surfaces) the excess pressure due to simple compression is $p_1 = B\left(\frac{\xi}{d}\right)$ in which B is the atmospheric pressure. From this must be subtracted the loss in pressure due to radial displacement (η) in order to obtain the excess pressure at any point in the film. For an annulus the radii of whose bounding cylinders are r and $r+dr$ the volume of contained air is

$$V_0 = 2\pi d \cdot r dr,$$

and the net loss in contained air due to radial displacement is

$$dV = \frac{\partial}{\partial r}(2\pi d \cdot r \cdot \eta) dr,$$

so that the resulting loss in pressure due to radial displacement is

$$p_2 = B\frac{dV}{V_0} = \frac{B}{r}\frac{\partial}{\partial r}(r\eta) = B\left(\frac{\eta}{r} + \frac{\partial \eta}{\partial r}\right). \tag{35}$$

The excess pressure at any point in the film is then $p = p_1 - p_2$. Since $\frac{dp}{dr}\left(= -\frac{dp_2}{dr}\right)$ is the pressure gradient, we have, analogously to Ohm's law, for the velocity of flow

$$\frac{\partial \eta}{\partial t} = -\frac{1}{R}\frac{\partial p}{\partial r} = \frac{1}{R}\frac{\partial p_2}{\partial r} \quad \text{whence} \quad \eta = \frac{1}{R}\int \frac{\partial p_2}{\partial r}dt, \tag{36}$$

in which R is a resistance coefficient, the inertia of the moving air being neglected as η is small. In treatises on fluid motion (Lamb, "Hydrodynamics," 4th ed., p. 576; also Drysdale, et als., "Mechanical Properties of Fluids," p. 116) it is shown [1] that for a fluid of viscosity μ flowing between parallel walls separated by a very small distance d,

$$R = \frac{12\mu}{d^2}. \tag{37}$$

Substituting in (35) the value of η from (36), and differentiating with respect to time, we have

$$\frac{R}{B}\frac{\partial p_2}{\partial t} = \frac{\partial^2 p_2}{\partial r^2} + \frac{1}{r}\frac{\partial p_2}{\partial r}, \tag{38}$$

or, since p_2 must vary as $e^{i\omega t}$, i.e., $\dot{p}_2 = i\omega p_2$,

$$\frac{\partial^2 p_2}{\partial r^2} + \frac{1}{r}\frac{\partial p_2}{\partial r} - \frac{i\omega R}{B}p_2 = 0. \tag{38a}$$

Here again on account of the axial symmetry of the structure, we encounter Bessel's equation, and the Bessel's functions of zero order are required for its solution. The pressure must be

[1] The resistance coefficient in this case is analogous to Poiseuille's coefficient for the case of a very narrow cylindrical tube. This latter is derived in Appendix A, eq. (I). See also problem 26, following Chapter III.

finite when $r = 0$, so as in § 12 we discard the function $K_0(kr)$ because it becomes infinite for $r = 0$. We thus have

$$p_2 = AJ_0(kr)e^{i\omega t} \quad \text{in which} \quad k^2 = -\frac{i\omega R}{B}, \tag{39}$$

and the complete solution is

$$p = p_1 - p_2 = \frac{B}{d}[\xi_0 - AJ_0(kr)]e^{i\omega t}. \tag{40}$$

In order to evaluate this expression, note the form of $J_0(kr)$ when $-k^2$ is imaginary. Letting $\alpha = \sqrt{\frac{\omega R}{B}}$, we have, instead of equation (25)

$$J_0(kr) = J_0(\sqrt{-i}\,\alpha r) = 1 + i\frac{\alpha^2 r^2}{2^2} - \frac{\alpha^4 r^4}{2^2 4^2} - i\frac{\alpha^6 r^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \tag{41}$$

or,

$$J_0(\sqrt{-i}\,\alpha r) = \text{ber}\,\alpha r + i\,\text{bei}\,\alpha r, \tag{41a}$$

using the terminology established by the electricians in solving the mathematically related problem of alternating-current resistance and inductance in a cylindrical conductor. (See A. Russell, *Phil. Mag.*, April, 1909, p. 524; or "Alternating Currents," V. I, Chap. VII). Tables have been computed for the ber and bei functions from the series

$$\left.\begin{aligned} \text{ber}\,\alpha r &= 1 - \frac{\alpha^4 r^4}{2^2 \cdot 4^2} + \frac{\alpha^8 r^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots \\ \text{bei}\,\alpha r &= \frac{\alpha^2 r^2}{2^2} - \frac{\alpha^6 r^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots \end{aligned}\right\}. \tag{41b}$$

These functions, and the corresponding functions of the second kind (ker αr and kei αr) are the key to the reactions in cylindrical structures when periodic driving forces are applied.

We began with the assumption that $\xi = \xi_0 \cos \omega t$. We must

therefore have $p_1 = \frac{B}{d}\xi_0 \cos \omega t$ and similarly retain only real quantities in the final solution. Writing for A, $Ce^{i\phi}$, and substituting (41*a*) in (40), for the real portion of the solution,

$$p = \frac{B}{d}\xi_0 \cos \omega t - C[\text{ber } \alpha r \cdot \cos(\omega t + \phi) - \text{bei } \alpha r \cdot \sin(\omega t + \phi)]. \tag{42}$$

To determine C and ϕ we note that, for all values of t, $p = 0$ when $r = a$ because of free communication with the atmosphere. Then, if

$$\text{ber } \alpha a = X \cos x, \quad \text{bei } \alpha a = X \sin x, \quad X^2 = \text{ber}^2\, \alpha a + \text{bei}^2\, \alpha a,$$

we must have

$$\frac{B}{d}\xi_0 \cos \omega t = CX \cos(\omega t + \phi + x);$$

hence,

$$\phi = -x, \quad \cos \phi = \frac{1}{X} \text{ber } \alpha a, \quad \sin \phi = -\frac{1}{X} \text{bei } \alpha a, \quad C = \frac{B\xi_0}{dX},$$

and the final solution is therefore

$$p = \frac{B\xi_0}{d}\left[1 - \frac{\text{ber } \alpha r \text{ ber } \alpha a + \text{bei } \alpha r \text{ bei } \alpha a}{\text{ber}^2\, \alpha a + \text{bei}^2\, \alpha a}\right] \cos \omega t$$

$$- \frac{B\xi_0}{d}\left[\frac{\text{ber } \alpha r \text{ bei } \alpha a - \text{bei } \alpha r \cdot \text{ber } \alpha a}{\text{ber}^2\, \alpha a + \text{bei}^2\, \alpha a}\right] \sin \omega t. \tag{42a}$$

In this solution for the pressure the coefficient of the *amplitude* $\xi_0 \cos \omega t$ represents the *stiffness per unit area* as a function of r. From (41*b*) ber (0) = 1 and bei (0) = 0; thus the stiffness at the center is $\frac{B}{d}\left[1 - \frac{\text{ber } \alpha a}{X^2}\right]$, falling to zero when $r = a$. The sine term is in phase with the *velocity* ($\dot{\xi} = -\omega\xi_0 \sin \omega t$), hence the *resistance per unit area* is $-\frac{1}{\omega}$ times the coefficient of $\xi_0 \sin \omega t$ in

(42*a*). The resistance is therefore $\frac{B}{\omega d} \cdot \frac{\text{bei } \alpha a}{X^2}$ at the center, and also falls to zero when $r = a$. Thus the vibrating piston with a variable distribution of impedance over its surface, should now be considered to bend slightly, if the argument is to proceed rigorously. This refinement, however, is beyond the practical necessities of the case, and we continue the discussion on the basis of the piston system of one degree of freedom.

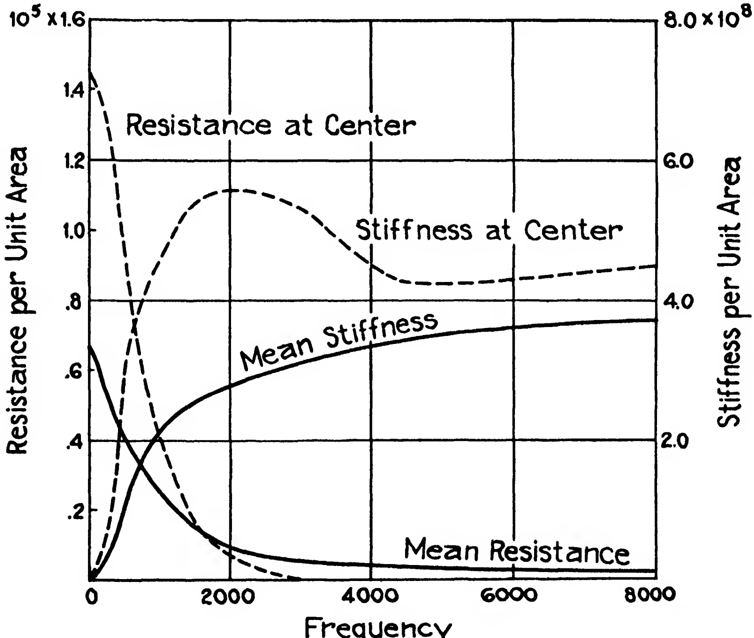


FIG. 5.—REACTIONS IN AIR DAMPING FILM: PISTON SYSTEM.

In Fig. 5 there are plotted curves showing roughly the frequency variations of resistance and stiffness *per unit area at the center* of the piston, and *mean* resistance and stiffness *per unit area*: all taken for a piston and damping disc of radius 1.63 cm. separated by an air film 2.2×10^{-3} cm. thick. From these curves a good idea can be gained of the typical reactions in the simplest air damped system. The data are based on some old

calculations for an early form of condenser transmitter,[1] the mean values of the constants being obtained by integration over the surface of the piston.

This theory has been tested, by application to a condenser transmitter of simple design, making certain modifications to allow for the shape of the membrane and it has been found that a good quantitative explanation of the behavior of the instrument is given. An interesting special test has also been made of the resistance formula for low frequencies, setting up a condenser transmitter [1] having a diaphragm weighted at the center with a heavy disc (to insure piston motion and lower the natural frequency), and measuring the quantity $\Delta = \frac{r_0}{2m}$ from an electrically obtained graph of the natural oscillations. In this case (since ber (0) = 1 and bei $(\alpha r) = \frac{1}{4}\alpha^2 r^2$ as $\alpha r \doteq 0$) we have, per unit area

$$r'_0 = \frac{1}{\omega}\frac{B}{d}\left[\frac{\alpha^2 a^2 - \alpha^2 r^2}{4}\right] = \frac{R}{4d}(a^2 - r^2), \tag{43}$$

and the total resistance, since $R = \frac{12\mu}{d^2}$, by (37), is

$$r_0 = 2\pi\int_0^a r'_0 \cdot r dr = \frac{3}{2}\frac{\pi\mu a^4}{d^3}, \tag{44}$$

or, per unit area, on the average

$$\bar{r}_0 = \frac{3}{2}\frac{\mu a^2}{d^3}. \tag{44a}$$

This gives the point at zero frequency for the mean resistance shown in Fig. 5.

[1] *Phys. Rev.*, XI, 1918, p. 449. The integrated values for *total stiffness* and *total resistance* over the piston are, respectively,

$$s = \frac{\pi a^2 B}{d}\left[1 - \frac{2}{\alpha a}\,\frac{\text{ber}\,\alpha a\,\text{bei}'\,\alpha a - \text{ber}'\,\alpha a\,\text{bei}\,\alpha a}{\text{ber}^2\,\alpha a + \text{bei}^2\,\alpha a}\right],$$

and

$$r' = \frac{2\pi a B}{\omega\alpha d}\left[\frac{\text{bei}\,\alpha a\,\text{bei}'\,\alpha a + \text{ber}\,\alpha a\,\text{ber}'\,\alpha a}{\text{ber}^2\,\alpha a + \text{bei}^2\,\alpha a}\right].$$

For the technology of condenser transmitter design, the reader may refer to recent papers by E. C. Wente [1] in addition to the reference already given. In recent practice the tendency has been to make use of the resistance properties of the air film to a high degree, and the problem has been considered here, in a simplified form because the instrument is an outstanding example of the application of viscosity damping to the vibrating system.

12. *Equivalent Piston; Mean Velocity; Diaphragms*

We now consider the principles and methods used in determining the area of the equivalent piston for an actual vibrating system. If the system radiates energy into the medium (i.e., into a compressible fluid) it is required to know the amount of fluid displaced in unit time in terms of the motion of the system. This constant, known as the "strength of the source" or the "rate of emission of fluid at the source" is best given as the product of a certain area and the mean or average velocity over that area: $A = \bar{\dot{\xi}} S$. If A is known we can compute the amount of radiation from the system and hence its radiation resistance. Both $\bar{\dot{\xi}}$ and S are, within reasonable limits, at our disposal, as long as their product is made equal to A.

In practical problems it is usually of advantage to adjust S so that, for purposes of calculation, the radiation conditions of the problem are most simply taken into account. For example, suppose a positively driven diaphragm of area S_1 is fitted to a tube of slightly larger cross-section S_2. In this case the sensible procedure is to replace the diaphragm (of unequal motion over its surface) by a piston of area S_2 and consider the piston to have a mean velocity $\bar{\dot{\xi}}_2 = \frac{S_1}{S_2}\bar{\dot{\xi}}_1$, in which $\bar{\dot{\xi}}_1$ is the mean velocity of the diaphragm over the area S_1. The area of the equivalent piston being so determined, the inherent mass, stiffness and resistance properties of the system may be related to the piston

[1] For these and other references to Acoustic Devices, see Appendix B.

in the way which best represents the internal structure of the system.

To determine A in a given case we must compute the mean velocity. A typical example of determining the mean velocity over the working area of a (simplified) telephone diaphragm will be given. The determination of the shape of a disc clamped at the edges, and vibrating in its normal modes, is a difficult problem, involving a differential equation of the fourth order and we shall only quote the result. It is shown in Rayleigh (§§ 217–221a) that the amplitude of such a disc is

$$\left.\begin{aligned} \xi &= C[J_0(kr) + \lambda J_0(ikr)] \cos (nt - \epsilon), \\ \text{in which} \quad k^2 &= \frac{n}{h}\sqrt{\frac{3\rho(1-\sigma^2)}{E}}. \end{aligned}\right\} \tag{45}$$

($2h$ = thickness; ρ = density; E = Young's modulus, σ = Poisson's ratio; $n = 2\pi$ times any natural frequency.)

At the boundary ($r = a$) there are two conditions to observe; both ξ and $\frac{d\xi}{dr}$ must vanish. These two conditions determine λ and k; and we have (assuming λ known)

$$\xi(kr) = \xi_0 \frac{J_0(kr) + \lambda J_0(ikr)}{1 + \lambda} \cos (nt - \epsilon). \tag{45a}$$

For the first natural frequency $k_1 a = 3.2$, $\lambda_1 = -\frac{J_0(ka)}{J_0(ika)} = .056$, the natural frequency being [cf. (45)],

$$f_1 = \frac{(3.2)^2\sqrt{E}\cdot h}{2\pi a^2\sqrt{3\rho(1-\sigma^2)}} = 2.95\frac{h}{\pi a^2}\sqrt{\frac{E}{\rho(1-\sigma^2)}}. \tag{46}$$

(Graphs of the Bessel's functions used; and of the shape of the diaphragm in this mode are given in Kennelly, "Electrical Vibration Instruments," pp. 304–305.) The average velocity, in terms of the maximum (central) velocity is:

$$\dot{\bar{\xi}} = \frac{2\pi}{\pi a^2}\int_0^a \dot{\xi}(kr)rdr = \frac{2\dot{\xi}_0}{k^2a^2}\int_0^{ka=3.2} \frac{J_0(kr) + \lambda J_0(ikr)}{1+\lambda} kr\cdot kdr. \tag{47}$$

The integral is readily calculated since

$$\int_0^b xJ_0(x)dx = xJ_1(x)\Big|_0^b.$$

The function $-iJ_1(ix)$ which appears on integrating the second term in the numerator is known as $I_1(x)$ and is a real quantity, if x is real; tables of $I_1(x)$ are available. The calculation (47) gives

$$\bar{\dot{\xi}} = .306\,\dot{\xi}_0, \tag{48}$$

which is the mean velocity sought. This value, applying to the *gravest symmetrical mode* is sufficiently accurate for many problems in which radiation from telephone diaphragms is considered, provided the frequency is less than twice the first natural frequency, and the vibrations are small.

The reader may compare the integration (47) with that given in Kennelly (loc. cit.), in which the square of the velocity is integrated over the diaphragm for the purpose of determining the mass constant. It may also be noted that the function

$$\xi = \xi_0\left(1 - \frac{r^2}{a^2}\right)^2 \qquad \left(\xi_0 = \frac{3}{128}\frac{(1-\sigma^2)}{E}\frac{Pa^4}{h^3}\right). \tag{49}$$

represents the static deflection under uniform load P per unit area, the elastic constants being taken as before. (See Love, Elasticity, Third Ed., 1920, p. 494.) If this function is integrated over the area, for the purpose of obtaining a mean static deflection we find $\bar{\xi} = \frac{1}{3}\xi_0$ [cf. eq. (20)]. Thus in terms of the velocity at the center, the mean velocity of the diaphragm does not vary greatly over the frequency range from zero to the first natural frequency. In view of certain departures from the theoretical mode of vibration in the actual form of vibrating diaphragms, due to lack of symmetry in the driving force, over driving, etc., it is not worth while to be too precise in the determination of such "constants" as the mean amplitude. The example has been given merely to illustrate the principle, and incidentally record a few facts concerning the vibration of a clamped circular plate.

The experimental study of diaphragms, i.e., of the variation of their constants under the constraints imposed in various structures belongs to the domain of engineering development and design with which we are not primarily concerned. The purpose of this chapter has been to present, in reasonably concrete terms, the essential properties of the simple vibrating system, and if this has been accomplished we have reached a point of departure.[1] In the next chapter a discussion will be given of general methods applicable to systems of several degrees of freedom. In later chapters we shall deal with the production and transmission of sound radiation.

PROBLEMS

1. (*a*) Given the following constants of a telephone diaphragm (in terms of motion at the center) for frequencies near resonance:

$$m = .675 \text{ gram.} \quad s = 2.67 \times 10^7 \text{ dynes/cm.}$$

$$r = 270 \text{ dyne. sec./cm.}$$

Determine the resonant frequency, f_0; the damping coefficient, Δ; the frequency of the natural oscillations.

(*b*) The center of the diaphragm is given an arbitrary displacement of .01 cm. from the equilibrium position and let go. Write the equation of motion.

(*c*) Write the equation of motion of the center of the diaphragm for an arbitrary initial velocity of 60 cm./sec., the system being started from the equilibrium position.

2. (*a*) A constant force F is suddenly applied to a simple vibrating system. Discuss the growth of the displacement with time for the *periodic* case. Find the value of t for the first point of inflection (or

[1] Some readers may desire at this point to consider in detail such matters as the elastic constants of solids, and the vibrations of bars and plates. For these, the references are to the classical theory, e.g., Lamb, Chaps. IV and V. It is planned to avoid, wherever possible, mere restatement of material already available.

point of zero acceleration) of the growth curve; show also that at that point

$$qs = r\dot{\xi} \qquad \text{if} \qquad q = \frac{F}{s} - \xi.$$

(*b*) A constant force F is suddenly removed from a *critically damped* (i.e., *aperiodic*) system. Discuss the decay of the displacement, noting that the two roots of the auxiliary equation are equal in this case, and the solution must be in the form

$$\xi = (A' + B't)e^{-\Delta t}.$$

Determine A' and B'; is there any point of inflection in the decay curve?

3. A force $\Psi_0 \cos \omega t$ acts on the diaphragm. Considering the equivalent simple system to have the constants as given in Ex. 1, find the r.m.s. values of the amplitude and velocity at resonance, for $\Psi_0 = 10$ dynes. Find the r.m.s. values for the amplitude and velocity for the frequencies 30 per cent above and below resonance; for the frequencies $\frac{1}{2\pi}(n \pm \Delta)$; also for zero frequency. Plot all these data to show the frequency response of the system.

4. Discuss analytically the motion of the central point of the diaphragm, when following steady state conditions the driving force $\Psi_0 \cos \omega t$ is suddenly removed.

5. Taking the velocity as $\dot{\xi} = \frac{\Psi_0}{Z}e^{i\omega t}$ in the steady state theory of the simple system, show that the instantaneous rate of dissipation in the system is $\dot{\xi}^2 r$.

6. Derive the simplest expression you can for the energy stored in the simple system, in the steady state. Does this fluctuate? Is the maximum kinetic energy ever equal to the maximum potential energy, and if so, when?

7. Derive equation (19).

8. A circular membrane of sheet steel, 4 cm. in diameter, and .005 cm. thick is stretched to a certain tension. Under a uniform static pressure of half an atmosphere applied to one side of the membrane,

the central deflection is .01 cm. What are the first three natural frequencies for the normal symmetrical mode of vibration?

9. A pistonlike air-damped system was set up with piston of radius 1.63 cm. and mass 42.9 grams. The observed damping constant of the system was 3.4×10^3, for a mean separation of 2.9×10^{-3} cm. between piston and damping plate. From these data what would you determine for the viscosity constant of the damping fluid, on the basis of low frequency theory?

10. Show that for low frequencies, the mass coefficient for a circular plate, clamped at the edge, is, in terms of the motion at the center, $\frac{1}{5}$ of the total mass of the free portion of the plate. Show also that the corresponding stiffness coefficient, for low frequencies, is

$$s = \frac{128}{9} \frac{\pi E h^3}{(1 - \sigma^2)a^2}, \qquad \text{[cf. (49)]},$$

and on this basis show that the first natural frequency of the plate is

$$f_1 = 2.98 \frac{h}{\pi a^2} \sqrt{\frac{E}{\rho(1 - \sigma^2)}},$$

which is closely equivalent to the more rigorously obtained result of (46).

CHAPTER II

General Theory of Vibrating Systems; Resonators and Filters

20. *Generalized Coordinates*

In a system of several degrees of freedom, that is, a system in which more than one independent variable (ξ_1, ξ_2, ξ_3 . . .) is required in order to adequately describe the motion, it is generally impossible to deduce the equations of motion from the Energy Principle, as was done in § 2 for case of a single degree of freedom. The reason for this is, that while the energy principle is necessarily valid, it can give no information regarding interchanges of energy between the inner parts of the system on which external forces do not act directly. The equations of motion of the system must take account not only of the external forces, but also of the mutual reactions within the system.

To begin with, the independent variables (ξ_1 . . . ξ_m) represent only those displacements which are allowed to take place, by virtue of the constraints inherent in the system. If we proceeded according to the Newtonian Method, the forces due to all these constraints would be resolved each into 3 rectangular components and, with the applied forces similarly resolved, we should require $3m$ ordinary mechanical equations to discuss the motion. But if we resolve the applied forces, so that only those components each of which acts in the direction of a possible displacement ξ_j are considered, and if we describe the constitution of the system only in terms of these displacements (ξ_1 . . . ξ_m) and of certain mass, stiffness and resistance factors which take into account the internal reactions, then by the application of one of the broader dynamical principles (such as that of D'Alembert, or of Least Action) it is possible to obtain

m equations which fully state the motion of the system. The particular machinery which we shall use to bring all this about is the method of Lagrange; the simplified coordinates ($\xi_1 \dots \xi_m$) are known as generalized coordinates; the forces [1] which tend to increase these displacements are called generalized forces; and as a result of the application of the method, the equations of motion are almost automatically obtained in concise and convenient form.

This method is fundamental to mechanics generally and indepensable in dealing with. the typical problems presented in vibrating apparatus. While we cannot take time to derive it, we can become familiar with its mechanism and its underlying principles. This we propose to do by an inductive method, first considering an easy problem in two degrees of freedom.

21. *System of Two Degrees of Freedom; Natural Oscillations in General*

A thin string of length $3l$, stretched to tension τ, has attached to it equal masses at two points distant l from either end. The masses (1) and (2) are allowed to vibrate in a vertical plane; their displacements from the equilibrium position of the system are respectively ξ_1 and ξ_2. Friction is taken into account by equal resistance constants r in each degree of freedom. The component of force on either particle, due to tension in the end-portion of the string and resolved in the direction of motion is $\frac{\tau}{l}\xi$. It is evident that there are two types of motion possible in the system, i.e., one with the displacements ξ_1 and ξ_2 having the same sign, and the other with ξ_1 and ξ_2 of opposite sign. In either case the component of the force on the first particle (for example), due to the tension in the mid-portion of the string, and resolved in the direction of ξ_1 is $\frac{\tau}{l}(\xi_1 - \xi_2)$. Letting $c = \frac{\tau}{l}$

[1] Strictly, the generalized force Ψ_i corresponding to any generalized coordinate ξ_i is defined as the ratio of the work done to the displacement, when ξ_i is varied by an infinitesimal amount $\delta\xi_i$, all of the other coordinates remaining constant.

the equations of motion are therefore

$$\left.\begin{aligned} m\ddot{\xi}_1 + r\dot{\xi}_1 + 2c\xi_1 - c\xi_2 &= \Psi_1, \\ m\ddot{\xi}_2 + r\dot{\xi}_2 + 2c\xi_2 - c\xi_1 &= \Psi_2, \end{aligned}\right\} \tag{51}$$

in which Ψ_1, Ψ_2, are the impressed forces acting on particles 1, 2, in their respective directions of motion. It appears that the interest in the problem will center chiefly on the effect of the elastic connection or "coupling" between the two vibrating masses.

The complete solution, as in the case of one degree of freedom, must contain expressions for both steady state and transient phenomena. Considering the latter first, and assuming motions of the type $\xi = \xi_0 e^{\lambda t}$ we have, since $\Psi_1 = \Psi_2 = 0$ for free oscillation,

$$\left.\begin{aligned} \beta\xi_1 - c\xi_2 &= 0, \quad (\beta = \lambda^2 m + \lambda r + 2c) \\ - c\xi_1 + \beta\xi_2 &= 0. \end{aligned}\right\} \tag{52}$$

In these two simultaneous equations, for consistency, we must have

$$\left.\begin{aligned} \begin{vmatrix} \beta, & -c \\ -c, & \beta \end{vmatrix} = 0 \quad &\text{or} \quad (\beta - c)(\beta + c) = 0, \\ &\text{or} \quad \beta_1 = +c,\ \beta_2 = -c. \end{aligned}\right\} \tag{53}$$

We thus obtain two equations in λ, namely

$$\lambda^2 m + \lambda r + c = 0 \quad \text{and} \quad \lambda^2 m + \lambda r + 3c = 0, \tag{54}$$

whose solutions are

$$\left.\begin{aligned} \lambda_1 &= -\frac{r}{2m} \pm i\sqrt{\frac{c}{m} - \frac{r^2}{4m^2}} = -\Delta \pm in'_1, \\ \lambda_2 &= -\frac{r}{2m} \pm i\sqrt{\frac{3c}{m} - \frac{r^2}{4m^2}} = -\Delta \pm in'_2, \end{aligned}\right\} \tag{54a}$$

taking $n' = \sqrt{n^2 - \Delta^2}$ as in §3. Now it is apparent that there are two types of oscillation which are normal for the system, i.e., one of frequency $f_1 = \frac{1}{2\pi}\sqrt{\frac{c}{m}}$ and one of frequency $f_2 = \sqrt{3}f_1$.

On account of the symmetry there is no reason for associating one of these motions more closely with particle (1) than with the particle (2); and it must be that both types of motion are simultaneously present in both degrees of freedom. Thus for the transient oscillations we must have (cf. eq. 6′),

$$\left.\begin{aligned} \xi_1 &= A_1 e^{-\Delta t} \sin (n'_1 t + \theta_1) + A_2 e^{-\Delta t} \sin (n'_2 t + \theta_2), \\ \xi_1 &= A'_1 e^{-\Delta t} \sin (n'_1 t + \theta'_1) + A'_2 e^{-\Delta t} \sin (n'_2 t + \theta'_2). \end{aligned}\right\} \quad (55)$$

Now all the 8 constants are not independent; only 4 are required for the general solution of 2 simultaneous second order equations. Letting $\beta_1 = c$, which corresponds to the lower-frequency oscillation $\left(f_1 = \frac{1}{2\pi} n'_1\right)$ we must have $\xi_1 = \xi_2$ (eq. 52) and similarly if $\beta_2 = -c$, for the other mode, $\xi_1 = -\xi_2$. The only solution consistent with these conditions is

$$\left.\begin{aligned} \xi_1 &= [A_1 \sin (n'_1 t + \theta_1) + A_2 \sin (n'_2 t + \theta_2)] e^{-\Delta t}, \\ \xi_2 &= [A_1 \sin (n'_1 t + \theta_1) - A_2 \sin (n'_2 t + \theta_2)] e^{-\Delta t}, \end{aligned}\right\} \quad (55a)$$

the numerical equivalence of the coefficients being due, of course, to the physical symmetry in the system. $A_1, A_2, \theta_1, \theta_2$, are to be determined in terms of the initial displacements and velocities assigned to the two masses (1) and (2).

Important conclusions which may now be reached, for this symmetrical system of two degrees of freedom are: (1) that the squares of the two natural frequencies are harmonically disposed with respect to the square of the natural frequency of either particle when the other is held in its equilibrium position, and (2) all possible free oscillations of the system are accounted for by compounding one motion in which the two particles are strictly in phase, and another in which they are strictly opposed in phase throughout the course of the motion; thus the *amplitudes* of corresponding transients in the two degrees of freedom are numerically equal.

For the more general system of m degrees of freedom we conclude: (cf. Lamb, p. 44).

1. There are m natural frequencies, and hence m component natural oscillations occurring simultaneously in all parts of the system. These are the "normal" oscillations of the system.

2. To the m natural frequencies correspond m damping factors ($\Delta_1 \ldots \Delta_m$) and these natural frequencies and damping factors are given by a determinantal equation of the mth order in λ^2, analogous to (53) in the foregoing example.

3. In each of the m modes of vibration the system oscillates exactly as if it had only one degree of freedom, the coordinates ($\xi_1, \xi_2 \ldots \xi_m$) having *constant ratios* as far as *amplitude* is concerned; the only arbitrary constants being the scale of the amplitude, and the phase constants.

The general equation embodying these conclusions, for the transients in the rth degree of freedom is:

$$\xi_r = \sum_{j=1}^{m} B_j A_{rj} e^{-\Delta_j t} \sin (n'_j t + \theta_j), \tag{55b}$$

in which the $2m$ constants B_j and θ_j are arbitrary, or assigned factors, dependent on initial velocities and amplitudes; while the ratios of the constants A_{rj} are determinate, since they depend on the equations

$$\left.\begin{array}{llll} \left.\dfrac{\xi_2}{\xi_1}\right|_{n=n_1} = \dfrac{A_{21}}{A_{11}}, & \left.\dfrac{\xi_2}{\xi_1}\right|_{n=n_2} = \dfrac{A_{22}}{A_{12}} & \ldots & \left.\dfrac{\xi_2}{\xi_1}\right|_{n=n_m} = \dfrac{A_{2m}}{A_{1m}} \\ \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot & \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ \left.\dfrac{\xi_m}{\xi_1}\right|_{n=n_1} = \dfrac{A_{m1}}{A_{11}} & \ldots \quad \cdot & \cdot \quad \cdot & \left.\dfrac{\xi_m}{\xi_1}\right|_{n=n_m} = \dfrac{A_{mm}}{A_{1m}} \end{array}\right\}. \tag{56}$$

We can take $A_{11} = A_{12} = \ldots A_{1m} =$ unity. For example, in (55a) we have $B_1 = A_1$, $B_2 = A_2$, $A_{11} = A_{12} = 1$, $A_{21} = 1$, $A_{22} = -1$.

22. *Steady State Theory, for Two Degrees of Freedom and in General*

To determine the behavior of the system in the steady state, and so complete the solution, we let one particle be driven by

a force $\Psi_1 = \Psi e_0^{i\omega t}$ (eq. 51), and simplify the procedure by taking $\Psi_2 = 0$. This entails no sacrifice in essentials, on account of symmetry. We then have, for motions of the type $\xi_1 = \xi_{01} e^{i\omega t}$ $\xi_2 = \xi_{02} e^{i\omega t}$,

$$\left.\begin{aligned}(-m\omega^2 + ir\omega + 2c)\xi_1 - c\xi_2 &= \Psi_0 e^{i\omega t},\\ -c\xi_1 + (-m\omega^2 + ir\omega + 2c)\xi_2 &= 0.\end{aligned}\right\} \tag{57}$$

The form of these equations shows at once the simplification to be gained, if we speak in terms of impedance and velocity, as in the dynamical theory of electrical networks. The set (57) is equivalent to

$$\left.\begin{aligned}\alpha\dot{\xi}_1 - \frac{c}{i\omega}\dot{\xi}_2 &= \Psi_0 e^{i\omega t},\\ -\frac{c}{i\omega}\dot{\xi}_1 + \alpha\dot{\xi}_2 &= 0,\end{aligned}\right| \tag{57a}$$

if
$$\alpha = r + i\left(m\omega - \frac{2c}{\omega}\right);$$

and, solving for the velocities, we have

$$\dot{\xi}_1 = \frac{\alpha\Psi_0 e^{i\omega t}}{\left(\alpha - \frac{c}{i\omega}\right)\left(\alpha + \frac{c}{i\omega}\right)} \quad \text{and} \quad \dot{\xi}_2 = \frac{\frac{c}{i\omega}\Psi_0 e^{i\omega t}}{\left(\alpha - \frac{c}{i\omega}\right)\left(\alpha + \frac{c}{i\omega}\right)}, \tag{58}$$

or, letting

$$\left(\alpha - \frac{c}{i\omega}\right) = \alpha_1 = r + i\left(m\omega - \frac{c}{\omega}\right) \quad \text{and} \quad \alpha_2 = r + i\left(m\omega - \frac{3c}{\omega}\right),$$

$$\dot{\xi}_1 = \frac{\alpha}{\alpha_1\alpha_2}\Psi_0 e^{i\omega t} \quad \text{and} \quad \dot{\xi}_2 = \frac{c}{i\omega}\cdot\frac{1}{\alpha_1\alpha_2}\Psi_0 e^{i\omega t}. \tag{58a}$$

These equations might be rationalized and studied in detail for the case a real applied force $\Psi_0 \cos \omega t$, acting on particle (1). Practically, in view of the similar study made in (§ 7) for the simple system this laborious operation is not necessary, in order to understand the behavior of the system. Note first

the quantities of impedance dimensions, giving them special designations:

$$Z_{11} = \frac{\alpha_1 \alpha_2}{\alpha} = \frac{D}{\alpha}; \quad Z_{12} = \frac{i\omega}{c}\alpha_1\alpha_2 = \frac{i\omega}{c}D. \tag{59}$$

The first may be called a "driving-point impedance" or "apparent impedance at the driving point"; the second a "transfer impedance" or "apparent impedance at point (2) for force applied at point (1)." Both Z_{11} and Z_{12} have minima for the two natural frequencies (cf. β_1, β_2) found in the "transient" solution, § 21; at these frequencies very large forced oscillations occur. At the intermediate frequency for which $\beta = 0$, $\alpha = r$; Z_{11} is a maximum and particle (1) moves with small amplitude; but at this frequency particle (2) has a larger amplitude, in the ratio $\frac{c}{i\omega r}$. c may be considered a "coupling factor," increasing with increasing tension in the string, or decreasing distance l between the particles. Thus for the frequency range between $f_1 = \frac{1}{2\pi}\sqrt{\frac{c}{m}}$ and $f_2 = \sqrt{3} f_1$ the apparatus, driven at point (1), and with amplitude or velocity observed at point (2) behaves like a crude filter, the *sharpness* of declines in amplitude for $f < f_1$ and $f > f_2$, and the height of the minimum amplitude *between* f_1 and f_2 both being increased as the resistance constant r is diminished; the intermediate minimum amplitude being farther increased with increasing coupling factor c. The general properties of the analogous electrical system, consisting of two identical meshes, coupled through a mutual capacity, are so well understood that we shall not pursue this limited problem further.

Some general deductions with regard to the steady state behavior of the system of m degrees of freedom are now warranted.

(1) When a periodic force of type $\Psi_0 \cos \omega t$ acts on any part of the system, every part of the system executes a vibration of the same period, the velocities and amplitudes of which are given by equations of the type (58*a*). In these equations there always appears in the denominator of the right-hand member

the same determinant $D(\alpha_1, \alpha_2, \ldots \alpha_m)$ which when placed equal to zero $[D(\beta_1 \cdot \beta_2 \ldots \beta_m)$, (eq. 53)] determines the natural frequencies and damping constants of the system. Thus maxima in the forced oscillations occur for those values of the quantities β_i which give $D(\alpha_1, \alpha_2, \ldots \alpha_m)$ minimum values.

(2) The relative phases and amplitudes of the motions in different parts of the system will vary as the frequency is varied. But if there were *no dissipation in the system*, the oscillations would *all be in constant phase relation.* [Note $\frac{\dot{\xi}_1}{\dot{\xi}_2} = \frac{\alpha}{c} i\omega$, (58*a*).]

(3) In general, for a system of m degrees of freedom, for a single force Ψ_j applied to the jth degree of freedom,

$$\dot{\xi}_k = \frac{M_k}{D}\Psi_j \quad \text{and} \quad \xi_k = \frac{M_k}{i\omega D}\Psi_j, \tag{60}$$

in which $D = \alpha_1\alpha_2 \ldots \alpha_m$, the determinant of the coefficients of the general equations corresponding to set (57), and M_k is the minor of D obtained by suppressing the kth column and the jth row of D, and multiplying by $(-1)^{j+k}$.

(4) The general solution of the problem is obtained by combining equations of the type (60) with corresponding equations of type (55*b*) and determining from the initial conditions the $2m$ arbitrary constants.

Finally, to the reader who has pondered the matter, it must be evident that since the determinant D plays such an important part in the analytical discussion of both transient and steady state conditions, it must provide a key wherewith to obtain equations (60) directly, if equations (55*b*) are known, and vice versa. This is indeed true: it must be so, in order to translate the intimate physical relations between steady state and transient phenomena in a given system into adequate mathematical terms. This was realized by Heaviside and the idea has been further developed by later students of oscillation theory. Those interested may refer to papers by J. R. Carson (*Phys. Rev.*, IX, 1917, p. 217; also the series on "Electric Circuit

Theory, etc.," *Bell Syst. Tech. Jour.*, 1925–1926) and T. C. Fry (*Phys. Rev.*, XIV, 1919, p. 117) for the analytical discussion, which is too specialized to include the present outline.

In view of what has just been said, it usually suffices to consider only the transient state of oscillation in order to understand the mechanism of a given system.

23. *Lagrange's Method and Equations of Motion*

To proceed with the general theory, consider first the Potential Energy, V. In mechanical problems it is usually an economy to derive such forces as depend only on the position of a body, by differentiating a certain Potential Energy function (if one exists) with respect to the coordinate in the direction of motion. Suppose that we had written, for the forces due to elastic reactions, in equation (51)

$$\left.\begin{aligned} -Q_1 &= \frac{\partial V}{\partial \xi_1} = 2c\xi_1 - c\xi_2, \\ -Q_2 &= \frac{\partial V}{\partial \xi_2} = 2c\xi_2 - c\xi_1. \end{aligned}\right\} \tag{61}$$

Integrating (61), we have

$$V = c\xi_1^2 - c\xi_1\xi_2 + \phi_2(\xi_2),$$

or equally

$$V = \phi_1(\xi_1) - c\xi_1\xi_2 + c\xi_2^2.$$

Since for consistency $\phi_1(\xi_1) = c\xi_1^2$ and $\phi_2(\xi_2) = c\xi_2^2$ there must be for this particular problem, a potential function

$$V = c(\xi_1^2 - \xi_1\xi_2 + \xi_2^2). \tag{61a}$$

The coefficients in the right-hand member are all numerically equivalent to c, due to the accident of equal spacing of the masses in loading the string.

Now consider the Kinetic Energy. For the discrete masses $m_1 = m_2$ of the problem we write at once

$$T = \tfrac{1}{2}m(\dot{\xi}_1^2 + \dot{\xi}_2^2), \tag{62}$$

there being no mutual reactions due to inertia. This is generally true in purely mechanical problems. (In the dynamical theory of electrical circuits mutual inertias (mutual inductances) do occur: the coupling between different degrees of freedom is accomplished more often in this way than otherwise). The advantage in setting up the quadratic expression for Kinetic Energy and obtaining the rate of change of momentum in any part of the system by successive differentiation of this function is evident. In the general case the homogeneous quadratic function of the velocities contains cross product terms $m_{ij}\dot{\xi}_i\dot{\xi}_j$ to allow for the mutual inertia reactions and we have for the force due to inertia, in any degree of freedom of the system

$$Q'_i = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}_i}\right). \tag{63}$$

To take account of frictional forces, or more correctly, of energy dissipated in friction Lord Rayleigh introduced another homogeneous quadratic function of the velocities, with resistance factors as coefficients, which function when differentiated with respect to a given velocity gives the frictional force against which work is done in the given degree of freedom when motion takes place. This function, known as the Dissipation Function, would be written for the simple problem we have been considering

$$F = \tfrac{1}{2}r(\dot{\xi}_1^2 + \dot{\xi}_2^2), \tag{64}$$

but in general it contains cross-product terms as well, though not in the problems we shall encounter. The function F is one-half the total rate of dissipation of energy in the system as a whole.

According to the method of Lagrange (ably presented by Rayleigh, I, Ch. IV, §§ 80–90), the equations of motion in terms of T, V and F are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\xi}_1}\right) + \frac{\partial F}{\partial \dot{\xi}_1} + \frac{\partial V}{\partial \xi_1} = \Psi_1, \tag{65}$$

and so on for the other coordinates. It is evident that, with T, V, and F known (eqs. 61*a*, 62, 64) for the problem of the preceding sections, we may by applying (65) obtain at once the equations of motion previously given (eq. 51).

In the classical notation, the three quadratic functions are written:

$$\left.\begin{aligned} T &= \tfrac{1}{2}(a_{11}\dot{\xi}_1^2 + a_{22}\dot{\xi}_2^2 + \ldots + a_{12}\dot{\xi}_1\dot{\xi}_2 + \ldots), \\ F &= \tfrac{1}{2}(b_{11}\dot{\xi}_1^2 + b_{22}\dot{\xi}_2^2 + \ldots + b_{12}\dot{\xi}_1\dot{\xi}_2 + \ldots), \\ V &= \tfrac{1}{2}(c_{11}\xi_1^2 + c_{22}\xi_2^2 + \ldots + c_{12}\xi_1\xi_2 + \ldots), \end{aligned}\right\} \qquad (66)$$

the various factors a_{jk}, b_{jk}, c_{jk} being respectively the mass, resistance and stiffness constants. In the case of the constants indicating coupling, $a_{jk} = a_{kj}$, etc.

From an algebraic standpoint it is usually possible to choose the coordinates so that $a_{jk} = b_{jk} = c_{jk} = 0$, eliminating cross-product terms; then, the new coordinates [1] $\xi'_1 \ldots \xi'_m$ become "normal" coordinates. These have interesting properties; the m equations of motion then contain each only a single variable, ξ_j, the mutual reactions between the different degrees of freedom being eliminated. One reason for the name "normal" is that, in replacing any actual coupled system by an equivalent normal system, the natural oscillations which occur separately in the different degrees of freedom of the normal system are of the same frequencies as the natural or normal oscillations which occur, simultaneously in all the degrees of freedom of the original system. The idea is of interest mathematically, and is occasionally of practical use as will appear in a later section. We note in passing that in studying the normal modes of a circular membrane (§ 10) the idea of normal coordinates is implied; one such coordinate being used for each mode of motion of any part of the membrane.

[1] See Whittaker, "Dynamics," Article 94, for practical details of the transformation; also Rayleigh, I, §§ 96–100.

24. *Resonators*

With the needs of the next section in mind, digress for an interval to consider resonators. In acoustics the term resonator has come to mean a simple vibrating system consisting of compressible fluid contained in an enclosure which communicates with the external medium through an opening of restricted area in one of the walls. Typical cases are illustrated in Figs. 6 and 6*a*. In either case, the stiffness in the system is due to the volumetric compression $-\frac{dV}{V_0}$ within the enclosure, the vibrating mass being the average quantity of fluid which surges to and fro in the neighborhood of the opening. The theory of resonators has been given in detail by Helmholtz (who invented them) and by Rayleigh: here only a simplified treatment is required.

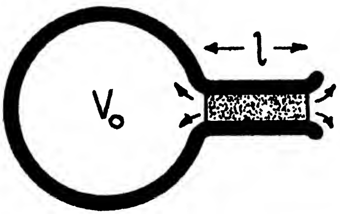


FIG. 6.

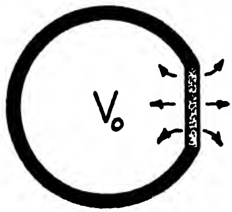


FIG. 6*a*.

RESONATORS.

Take first the case of the "bottle," Fig. 6. If ξ represents a small motion of the "plug" of air in the neck of the bottle, we have, for the force acting on the area S of the neck,

$$Q = -\gamma p_0 \frac{dV}{V_0} S = -\gamma p_0 \frac{S^2}{V_0} \xi = -\frac{\rho c^2 S^2}{V_0} \xi, \qquad (67)$$

in which $-\gamma p_0 \frac{dV}{V_0}$ is the increase in air pressure, originally p_0, due to adiabatic compression $-\frac{dV}{V_0}$; we borrow from § 30 of the following chapter the useful relation

$$\gamma p_0 = \rho c^2, \qquad (\kappa') \text{ from } (105a) \text{ and } (108)$$

in which ρ is the density and c the velocity of sound. The mass of the (nearly cylindrical) plug of moving air is ρSl, if l is the length of the neck. Thus we have for effective mass and elastic constants

$$a_1 = \rho S l, \qquad c_1 = \frac{\rho c^2 S^2}{V_0}, \tag{68}$$

the equation of motion being $a_1 \ddot{\xi} + c_1 \xi = 0$; whence

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{c_1}{a_1}} = \frac{c}{2\pi}\sqrt{\frac{S}{l \cdot V_0}} = \frac{c}{2\pi}\sqrt{\frac{K}{V_0}} \tag{69}$$

for the natural frequency. The quantity $K = \frac{S}{l}$ is called the *conductivity* of the neck, on account of its dimensions.

The natural frequency of any simple resonator is best given in terms of the "conductivity" of its aperture, its volume, and the constant c which depends on the medium in which the resonator is immersed. In the case of resonator of Fig. 6*a* with no neck there is an unequal distribution in fluid velocity at different points in the opening; the effective mass of the moving air in the circular aperture in the thin wall has been determined, in terms of a conductivity K as $\frac{\rho S^2}{K}$, the quantity K being equal to *the diameter of the opening*. (See Lamb §§ 82, 86, Rayleigh, Vol. II, § 306.) Now imagining an equivalent piston of area S and mean amplitude of motion ξ, (c_1 being unchanged) we obtain equation (69) for the natural frequency as before.

The notion of conductivity, in addition to leading to a simple formula for the natural frequency, facilitates the determination of total conductivity, when two or more conductivities are connected in series or in parallel; the formulae are identical with those used in computing electrical conductivities. Suppose for example it is desired to "correct" the conductivity formula for a short neck, to allow for the divergence of the air stream at the end of the neck: this is equivalent to adding the

reciprocal conductivity for the neck itself to that for the terminal orifices, for which, taken together, the formula $K=2r$ applies. In this case we have

$$K_1 = \frac{S}{l} = \frac{\pi r^2}{l}, \qquad K_2 = 2r,$$

$$K = \frac{K_1 K_2}{K_1 + K_2} = \frac{K_1}{1 + \dfrac{K_1}{K_2}} = \frac{\dfrac{\pi r^2}{l}}{1 + \dfrac{\pi r}{2l}}. \tag{69a}$$

These results will find application in the problem of acoustic filters.

Thus far no account has been taken of the influence of dis sipation on the behavior of the resonator. Dissipation due to radiation is usually of much greater importance than dissipation due to friction in the neck of the resonator, provided the resonator is so situated that it can radiate. (In a coupled filter system, for example, it is not designed to radiate.)

We have tacitly assumed in the preceding treatment that the resonator was small as compared with the wave length; it is therefore reasonable to make the same assumption with regard to the aperture. Making use again of results to be obtained later, we have, due to radiation,

$$b_1 = \frac{\rho \omega_0^2 S^2}{4\pi c} = \frac{c\rho K S^2}{4\pi V_0}; \tag{175}$$

and as

$$a_1 = \frac{\rho S^2}{K}, \qquad \Delta_1 = \frac{b_1}{2a_1} = \frac{cK^2}{8\pi V_0}. \tag{69b}$$

The form in which b_1 and Δ_1 are written here emphasizes their dependence on the dimensions of the resonator. The derivation of the formulae for b_1 and a_1 will be considered in Chapters III and IV. For the present the reader may merely note their dimensional correctness.

Viscosity damping, while serious only in the case of resona-

tors with small apertures, or long necks, should at least be considered. The mass of moving air in the latter case is ρlS. To find the frictional resistance we make use of a relation developed in Appendix A which embodies Helmholtz's coefficient for the resistance offered to the oscillations of a viscous fluid in a tube:

$$-\delta p \equiv p_x - p_{x+l} = +Rl\dot{\xi} \quad \text{in which} \quad R = \frac{1}{r}\sqrt{2\rho\mu\omega}, \qquad (69c)$$

$\frac{\delta p}{l}$ is the pressure gradient, r the radius of the tube, and μ the viscosity coefficient. For a tube of section S, we have,

$$\textit{force due to friction} = -S\,\delta p = RlS\cdot\dot{\xi},$$

and therefore

$$b_2 = RlS; \qquad (69d)$$

thus for the component of damping due to viscosity,

$$\Delta_2 = \frac{b_2}{2a_1} = \frac{RlS}{2\rho lS} = \frac{1}{r}\sqrt{\frac{\mu\omega}{2\rho}}. \qquad (69e)$$

It will be noted that dimensionally, the r in this formula merely offsets other linear dimensions in the definition of μ and ρ; the damping coefficient due to friction does not depend on the area S, except as undue constriction in $S = \pi r^2$ enhances friction according to Helmholtz's coefficient. Thus for n similar necks to a resonator, if all the motions were in phase, the ratio Δ_2 would be the same as for one alone; but of course the energy dissipated ($b_2\dot{\xi}^2$) would be proportional to n. Herein is suggested a means for varying the natural frequency of a resonator without changing its damping, if the damping is due entirely to friction in the neck; for the frequency will rise with S, or with the number of orifices used.

The *raison d'être* of a resonator is to select and amplify a sound component of given frequency, in the same way that an alternating current is selected and amplified by tuning an electrical circuit. (The classic example of this procedure is the work

of Helmholtz on the analysis of the vowel sounds, but there are important modern applications of resonators, as will be noted in Appendix B.) To illustrate, we shall show how the particle velocity and the excess pressure of an incident sound wave are amplified when the resonator is tuned to the frequency of the incident wave. As before, we borrow from Chapter III a necessary mechanical relation; there it is shown that for plane waves (§ 31)

$$\delta p = \rho c \cdot \dot{\xi}_1, \tag{117a}$$

in which δp is the excess pressure, $\dot{\xi}_1$ the particle velocity and ρc the radiation resistance of the medium.

The driving force on the mouth of the resonator is (to first approximation) $S \cdot \delta p\, e^{i\omega t}$, hence the velocity in the mouth of the resonator at resonance is $S \cdot \frac{\delta p\, e^{i\omega t}}{b_1}$; and if the opening is not so constricted that it brings viscosity into play, the resistance b_1 will be due entirely to radiation. Making use of equations (69), (175), and (117a) above, we have for the velocity in the orifice

$$\dot{\xi} = \frac{S\,\delta p}{b_1} e^{i\omega t} = \frac{4\pi V_0}{KS} \dot{\xi}_1 e^{i\omega t} = \frac{4\pi c^2}{\omega^2 S} \dot{\xi}_1 e^{i\omega t}, \tag{69f}$$

in which $\dot{\xi}_1$ is the maximum particle velocity of the incident sound wave. The quantity $\frac{4\pi c^2}{\omega^2 S} = \frac{\lambda^2}{\pi S}$, since it is the ratio $\frac{\dot{\xi}_{max.}}{\dot{\xi}_1}$ may be termed the *velocity amplification coefficient* of the resonator; and if we had a velocity measuring device (e.g., a hot wire microphone) set up in the orifice, we should expect a gain of this order of magnitude in using the resonator to amplify the original particle velocity of the wave. To illustrate, suppose that a resonator is tuned to 1000 cycles, and has an area of orifice $S = \frac{\pi}{4}$ sq. cm. (as in the problem of § 25). We then have, since $\frac{c}{\omega} = 5.3$, a velocity amplification ratio of 450, which would obviously be worth while. We remark that since the velocity

amplification varies inversely as the area of the orifice, small orifices usually imply losses due to viscosity, one sh not expect to realize in practice the large values of ampli tion which the simple theory would indicate.

Following Helmholtz, many have used the resonator a small tube leading from the back of the resonator to ear, so that the pressure changes within the resonator act tually, with little loss, directly on the ear. Hence the ception of *pressure-amplification*, that is the ratio of maxir pressure within the resonator to the maximum driving pres of the incident sound wave. Since the excess pressure in resonator is, from (67), $\frac{Q}{S}$, we have,

$$\delta p_r\Big|_{\text{max.}} = \frac{Q}{S}\Big|_{\text{max.}} = \frac{\rho c^2 S}{V_0 \omega}\dot{\xi}\Big|_{\text{max.}} = \frac{4\pi c^3}{V_0 \omega^3}\rho c\dot{\xi}_1 = \frac{4\pi c}{\omega K}\delta p, \quad (($$

putting $\dot{\xi}$ in terms of $\dot{\xi}_1$ from (69*f*). The quantity $\frac{4\pi c}{\omega K}$ is nov *pressure-amplification coefficient* of the resonator. In gen this is less than the velocity amplification coefficient, for quantities are in the ratio

$$\frac{\textit{Pressure Amplification}}{\textit{Velocity Amplification}} = \frac{\omega}{c}\frac{S}{K} = \frac{2\pi S}{\lambda K},$$

in which λ is the wave length of the sound. In the example viously used, from § 15 (since $\frac{\omega}{c} = \frac{1}{5\cdot 3}$ (at 1000 cycles) $\frac{S}{K} = \frac{\pi}{4}$) the pressure amplification is the less in the ratio c but it is evident that the resonator is still of advantage in plifying the sound as it affects the ear.

A problem in sound amplification for a resonator of degrees of freedom is set for the reader, at the end of this cl ter. Some further consideration of the behavior of a reson in a field of sound waves will be given in § 47, Chap. IV.

It must be noted that the simple theory of this section

plies only to the gravest mode of the typical resonator. In the higher modes the motions within the resonator are not all in phase, and various patterns of nodes and loops are set up within the enclosure. These depend on the shape of the enclosure as well as the frequency; the calculations become much more complicated and except in the case of organ pipes the higher modes are of little practical importance. The theory of organ pipes can rest until certain other problems have been considered.

25. *Resonator Coupled to a Diaphragm*

Consider now a problem in two degrees of freedom, one of the elements being a resonator. The system is sketched in Fig. 7; one wall of the cylindrical enclosure being a telephone

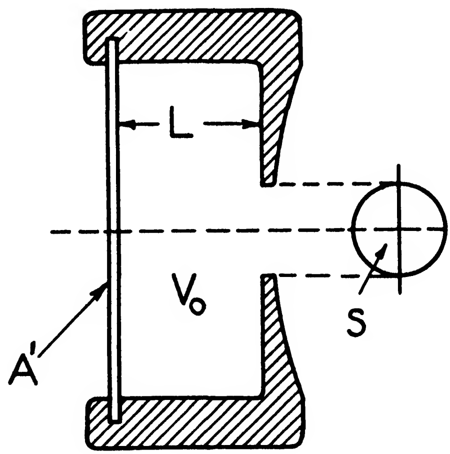


FIG. 7.—DIAPHRAGM AND COUPLED RESONATOR.

diaphragm—i.e., a clamped circular plate. We desire to find the effect of coupling the resonator to the diaphragm; to add interest to the problem we shall design the resonator beforehand so that its natural frequency, as a simple system, is equal to that of the diaphragm alone.

The following constants relate to the diaphragm: mass constant a_1 = .50 gram, in terms of velocity at center; area

= 15 sq. cm. = area of the cylinder section; average amplitude = .30 of amplitude at center (cf. § 12); f_0 = 1000, whence c_1 (central stiffness coefficient) = 1.97×10^7 c.g.s.; resistance (central) b_1 = 200 c. g. s.

For the resonator, take the length of the cylinder L = 1.86 cm.; the volume is then V_0 = 27.9 cu. cm. and for a diameter of orifice K = 1 cm. we shall have f_0 = 1000 since $c = 3.32 \times 10^4$ cm./sec. The constants of the resonator are then, since K = 1.0,

$$a_2 = \rho S^2 (= 8 \times 10^{-4}),$$

$$b_2 = \frac{\rho c S^2}{4\pi V_0} (= 7.6 \times 10^{-2}),$$

$$c_2 = \frac{\rho c^2 S^2}{V_0} (= 3.16 \times 10^4),$$

in which S is the area of the equivalent piston, or approximately the area of the orifice, $\frac{\pi}{4}$.

Now to deal with the diaphragm on an equivalent piston basis, let A' = 15 sq. cm. (area of diaphragm) and $A = (.3 \times 15) = 4.5$ sq. cm. Letting ξ_1 stand for motion of the center of the diaphragm, and ξ_2 stand for the motion in the orifice (piston) we have, for the excess pressure in the cylinder

$$p = \rho c^2 \frac{dV}{V_0} = \frac{\rho c^2}{V_0}(A\xi_1 - S\xi_2).$$

The expression in the parentheses is a net equivalent piston area multiplied by a certain amplitude, say $\bar{A} \cdot \bar{\xi}$. Letting this be a "normal coordinate" for the moment, the potential energy due to compression is

$$V = \int_0^{\bar{\xi}} p \cdot d(\bar{A}\bar{\xi}) = \frac{\rho c^2}{V_0} \int_0^{\bar{\xi}} (A\xi_1 - S\xi_2) d(A\xi_1 - S\xi_2)$$

$$= \frac{\rho c^2}{2V_0}(A\xi_1 - S\xi_2)^2.$$

Now adding the potential energy due to the diaphragm itself, we have for the total potential energy of the system,

$$V = \tfrac{1}{2}\left[c_1\xi_1{}^2 + \frac{\rho c^2}{V_0}(A^2\xi_1{}^2 - 2AS\xi_1\xi_2 + S^2\xi_2{}^2)\right],$$

or, if we take a new set of constants, viz.,

$$c_{11} = c_1 + k^2c_2,$$

$$c_{12} = kc_2,$$

$$c_{22} = c_2,$$

in which $k = \dfrac{A}{S} = 5.74$, we have

$$V = \tfrac{1}{2}(c_{11}\xi_1{}^2 - 2c_{12}\xi_1\xi_2 + c_{22}\xi_2{}^2). \tag{70}$$

The Kinetic Energy and Dissipation Functions are:

$$\left.\begin{aligned} T &= \tfrac{1}{2}(a_1\dot{\xi}_1{}^2 + a_2\dot{\xi}_2{}^2),\\ F &= \tfrac{1}{2}(b_1\dot{\xi}_1{}^2 + b_2\dot{\xi}_2{}^2).\end{aligned}\right\} \tag{70a}$$

T, V, and F are now in the form of (66) and applying the Lagrangian operation (65), letting applied forces equal zero, we have the following equations of motion for natural oscillations:

$$\left.\begin{aligned} a_1\ddot{\xi}_1 + b_1\dot{\xi}_1 + c_{11}\xi_1 - c_{12}\xi_2 &= 0,\\ a_2\ddot{\xi}_2 + b_2\dot{\xi}_2 + c_{22}\xi_2 - c_{12}\xi_1 &= 0.\end{aligned}\right\} \tag{71}$$

Following the usual method [§ 21, eq. (52)] the determinantal equation in λ takes the form

$$\left.\begin{aligned} D(\beta) &= \beta_1\beta_2 - c_{12}{}^2 = 0,\\ \text{in which}\qquad \beta_1 &= \lambda^2 a_1 + \lambda b_1 + c_{11},\\ \beta_2 &= \lambda^2 a_2 + \lambda b_2 + c_{22}.\end{aligned}\right\} \tag{72}$$

This gives the following equation in λ:

$$a_1a_2\lambda^4 + (a_1b_2 + b_1a_2)\lambda^3 + (b_1b_2 + c_{11}a_2 + c_{22}a_1)\lambda^2$$
$$+ (b_1c_{22} + b_2c_{11})\lambda + (c_{11}c_{22} - c_{12}{}^2) = 0. \tag{72a}$$

The development has been carried rigorously thus far as a matter of principle; but it is evident that, due to the essential asymmetry between the two degrees of freedom and due to the inclusion of dissipation constants, a rigorous solution of (72*a*) for the two pairs of roots λ_1, λ'_1; λ_2, λ'_2; is a laborious matter. The most practical method to follow to obtain a fair approximate solution of (72*a*) is to construct the equivalent equation from the roots

$$\lambda_1 = -\Delta_1 \pm in_1, \quad \lambda_2 = -\Delta_2 \pm in_2,$$

or

$$\lambda^4 + 2(\Delta_1 + \Delta_2)\lambda^3 + (4\Delta_1\Delta_2 + n_1^2 + n_2^2)\lambda^2 + 2(n_1^2\Delta_2 + n_2^2\Delta_1)\lambda + n_1^2n_2^2 = 0.$$

We thus have, for the natural frequencies (neglecting the products b_1b_2 and $\Delta_1\Delta_2$ as relatively small quantities) by comparison of coefficients,

$$\left.\begin{aligned} n_1^2 + n_2^2 &= \frac{c_{11}}{a_1} + \frac{c_{22}}{a_2} = n_{11}^2 + n_{22}^2, \\ n_1^2n_2^2 &= \frac{c_{11}c_{22}}{a_1a_2} - \frac{c_{12}^2}{a_1a_2} = n_{11}^2n_{22}^2 - n_{12}^4; \end{aligned}\right\} \tag{73}$$

and for the damping constants

$$\left.\begin{aligned} \Delta_1 + \Delta_2 &= \frac{b_1}{2a_1} + \frac{b_2}{2a_2} = \Delta_{11} + \Delta_{22}, \\ n_2^2\Delta_1 + n_1^2\Delta_2 &= \frac{c_{22}}{a_2}\cdot\frac{b_1}{2a_1} + \frac{c_{11}}{a_1}\cdot\frac{b_2}{2a_2} = n_{22}^2\Delta_{11} + n_{11}^2\Delta_{22}. \end{aligned}\right\} \tag{74}$$

The equations are placed in this form to emphasize the *redistribution or repartition* which takes place in the quantities Δ and n^2 when the two unequal simple systems are coupled together: a point which could not be made in the problem of (§ 21) because of the symmetry there in n_{11}^2 and Δ_{11} as compared with n_{22}^2 and Δ_{22}.

To better express the new natural frequencies in terms of the original natural frequencies (n_{11}^2, n_{22}^2) we let $n_1^2 + n_2^2 = 2A$ and

$n_1^2 n_2^2 = B$ in (73), and eliminating one of these variables we have the quadratic in n^2

$$n^4 - 2An^2 + B = 0,$$

whence n_1^2, n_2^2 are the roots

$$n^2 = A \pm \sqrt{A^2 - B}$$

$$= \frac{n_{11}^2 + n_{22}^2}{2} \pm \sqrt{\frac{(n_{11}^2 + n_{22}^2)^2}{4} - (n_{11}^2 n_{22}^2 - n_{12}^4)}. \quad (73a)$$

When n_1^2 and n_2^2 are found, the simultaneous equations (74), are solved, giving for the new damping constants

$$\left.\begin{aligned} \Delta_2 &= \frac{n_{22}^2 \Delta_{11} + n_{11}^2 \Delta_{22} - (\Delta_{11} + \Delta_{22}) n_2^2}{n_1^2 - n_2^2}, \\ \Delta_1 &= \Delta_{11} + \Delta_{22} - \Delta_2. \end{aligned}\right\} \quad (74a)$$

After a tedious computation, according to (73a) and (74a) we have the following values for the system of diaphragm and coupled resonator:

$$f_1 = 1120, \quad \Delta_1 = 131.0; \quad f_2 = 885, \quad \Delta_2 = 116.5;$$

the original values for diaphragm alone being

$$f_0 = 1000, \quad \Delta = 200,$$

and for the resonator alone $f_0 = 1000$, $\Delta = 47.5$. It may be noted that by simply coupling the volume V_0 (without resonator properties, i. e., with the orifice closed) to the diaphragm, the stiffness constant of the diaphragm would have been raised from $c_1 (= 1.97 \times 10^7)$ to $c_{11} (= 2.07 \times 10^7)$ i.e., only about 5 per cent, thus increasing the natural frequency of the diaphragm with this added constraint only to the value $\frac{n_{11}}{2\pi} = 1022$, or about 2 per cent. It appears then, that a very considerable change in natural frequencies, and a more equal distribution of damping in the system has been brought about by coupling to the heavy diaphragm a relatively light resonator system. The secret of

this effect is of course the kinetic energy residing in the mouth of the resonator, where very rapid motion of a relatively small mass takes place.

The study of the steady state behavior of the system is one of the problems at the end of this chapter.

26. *The Problem of the Loaded String; Filters*

The problem of a vibrating string of beads, that is, a long tense string loaded with equal masses, equally spaced, occupies a key position in oscillation mechanics. In it, the general theory reaches a climax, and from it have come important technical applications, such as the periodic electrical structures or filters [1] which are indispensable in telephony. The essentials of the theory, neglecting dissipation, can be quickly given in view of the preceding developments.

The length of the string is $(m + 1)l$; its tension τ; the m masses, a_1 each. The kinetic energy is then

$$T = \tfrac{1}{2}a_1(\dot{\xi}_1^2 + \dot{\xi}_2^2 + \ldots \dot{\xi}_m^2). \tag{75}$$

[1] The invention of the Electric Wave Filter is due to G. A. Campbell and is embodied in his U. S. Patent No. 1,227,113, dated May 22, 1917. Lagrange gave in the "Mécanique Analytique" the first general solution to the problem of the loaded string. Routh ("Advanced Rigid Dynamics," § 411) after discussing Lagrange's solution, points out that there may be a period of excitation of the string which is "so short that . . . no motion of the nature of a wave is transmitted along the string." Rayleigh (I, §148*a*) investigated the question of wave reflection, at the transition point between two strings of equal tension, but of different loading, and derives an equation for the ratio of reflected to incident amplitude, which is virtually the same as the equation for optical reflection, and identical with eq. (120), §31, which we shall encounter in dealing with acoustic reflection. The analogy between the mechanical problem and that of optical dispersion has been dealt with by various writers from the time of Stokes. A paper by C. Godfrey (*Phil. Mag.*, 45, 1898, p. 356) deals with the propagation of waves along the loaded string, and its optical analogy; and one by J. H. Vincent (*Phil. Mag.*, 46, 1898, p. 557) describes a dispersion model (a periodically weighted spring) which has a definite frequency limit of Wave Transmission. Finally there is a paper by H. Lamb (*Mem. Manch. Lit. and Phil. Soc.*, 42, No. 3, 1898, p. 1) on waves in a medium having a periodic discontinuity of structure. In a one-dimensional medium of this character, Lamb found that there were very definite selective properties, for waves of certain frequencies, and in conclusion he points out that "a dynamically equivalent problem is that of the propagation of sound waves along a tube having a series of equidistant bulbous expansions . . . ."

For small displacements, the forces on the first and last particles are respectively (cf. § 21),

$$2c_1\xi_1 - c_1\xi_2 \quad \text{and} \quad 2c_1\xi_m - c_1\xi_{m-1} \qquad \left(c_1 = \frac{T}{l}\right)$$

the force on the rth particle being

$$c_1[(\xi_{r-1} - \xi_r) + (\xi_{r+1} - \xi_r)],$$

and it is evident that these forces can be derived from a potential energy function (cf. eqs. 61, 61*a*),

$$V = \tfrac{1}{2}c_1[\xi_1^2 + (\xi_2 - \xi_1)^2 + (\xi_3 - \xi_2)^2 + \ldots + \xi_m^2]. \quad (76)$$

For the natural oscillations we have the m equations of motion:

$$\left.\begin{aligned} a_1\ddot{\xi}_1 + c_1(\xi_1 - 0 + \xi_1 - \xi_2) &= 0, \\ a_1\ddot{\xi}_2 + c_1(\xi_2 - \xi_1 + \xi_2 - \xi_3) &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad &\cdot \\ a_1\ddot{\xi}_m + c_1(\xi_m - \xi_{m-1} + \xi_m - 0) &= 0. \end{aligned}\right\} \quad (77)$$

Now assuming motions of the type $e^{\lambda t}$, $(\ddot{\xi}_r = \lambda^2\xi_r)$, and letting $C = 2 + a_1\dfrac{\lambda^2}{c_1}$, we have

$$\left.\begin{aligned} C\xi_1 - \xi_2 + 0 + \quad \cdot \quad \cdot \quad &= 0, \\ -\xi_1 + C\xi_2 - \xi_3 + \quad \cdot \quad \cdot \quad &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad 0 - \xi_{m-1} + C\xi_m &= 0, \end{aligned}\right\} \quad (78)$$

the determinantal equation of the coefficients being

$$D_m = \begin{vmatrix} C, & -1, & 0 & \cdot & \cdot \\ -1, & C, & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1, & C, & -1 \\ \cdot & \cdot & 0, & -1, & C \end{vmatrix} = 0, \quad m \text{ rows.} \quad (79)$$

There are of course m roots in $\lambda^2 \equiv -n^2$, if the equation $D_m = 0$

is solved; the procedure is theoretically the same as in the problems of a few degrees of freedom we have already solved. But, on account of algebraic difficulty a special device is applied.

By analogy from the problem of the string of length $3l$, loaded with two particles, we infer that of the m modes, the gravest will be with all the masses in phase, and as the frequency of the mode rises, the string will show an increasing number of nodes and loops until in the mode of highest frequency each oscillating particle will be exactly out of phase with its nearest neighbors. It can thus be assumed that the spacing between the natural frequencies will be harmonic (or nearly so) for the lower frequencies, while in the upper range, up to the highest natural frequency, the spacing will become closer and closer. A spectrum band is a fair analogy, with the component lines becoming very close to each other as the head of the band is reached.

To obtain the natural frequencies from the roots of $D_m(C) = 0$ we make use of a well-known trigonometric substitution which is obtained as follows (Webster, "Dynamics," p. 166): $D_m(C)$ is expanded in terms of its first minors, thus,

$$\left.\begin{aligned} D_m &= CD_{m-1} - D_{m-2}, \\ \text{or} \quad CD_{m-1} &= D_m \quad + D_{m-2}, \end{aligned}\right\} \tag{80}$$

and this suggests the trigonometric relation

$$2A \sin m\theta \cos \theta = A[\sin (m+1)\theta + \sin (m-1)\theta] \tag{81}$$

provided the constant A is correctly determined; the transformation being from the variable C to θ. Now, if $C = 2 \cos \theta$, we must have $D_m = A \sin (m+1)\theta$, by comparison of (81) with (80); and since $D_1(C) = C = 2 \cos \theta$,

$$D_1(C) = A \sin (1+1)\theta = 2A \sin \theta \cos \theta,$$

or $A = \dfrac{1}{\sin \theta}$, and finally

$$D_m(C) = \frac{\sin (m+1)\theta}{\sin \theta}, \quad \text{in which} \quad C = 2 \cos \theta. \tag{82}$$

For $D_m(C)$ to vanish, θ must equal $\frac{k\pi}{(m+1)}$, (k being any integer), or

$$2 + \lambda^2 \frac{a_1}{c_1} \equiv C = 2 \cos \frac{k\pi}{m+1}, \tag{83}$$

and since $\cos 2x = 1 - 2 \sin^2 x$,

$$1 + \frac{\lambda^2 a_1}{2c_1} = 1 - 2 \sin^2 \frac{k\pi}{2(m+1)}, \tag{84}$$

or

$$-\lambda^2 = n^2 = 4\frac{c_1}{a_1} \sin^2 \frac{k\pi}{2(m+1)}, \tag{85}$$

which gives for the natural frequencies (letting $k = 1, 2, \ldots m$)

$$f_1 = \frac{1}{\pi}\sqrt{\frac{c_1}{a_1}} \sin \frac{1}{m+1}\frac{\pi}{2}, \quad \ldots \quad f_m = \frac{1}{\pi}\sqrt{\frac{c_1}{a_1}} \sin \frac{m}{m+1}\frac{\pi}{2}. \tag{85a}$$

Thus the natural frequencies are proportional to the abscissae or ordinates in a quadrant divided into $(m+1)$ equal parts,

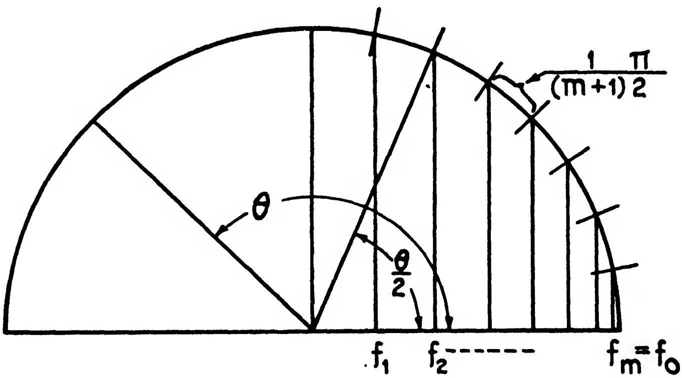


FIG. 8.—SPACING OF RESONANT FREQUENCIES, LOADED STRING.

Fig. 8. If the number of particles, or sections of the string is very large, we can write for the highest natural frequency

$$f_m \equiv f_0 = \frac{1}{\pi}\sqrt{\frac{c_1}{a_1}}. \tag{85b}$$

The general solution for the natural oscillations is of the type of equation (55*b*); thus neglecting dissipation

$$\xi_r = \sum_{k=1}^{m_1} B_k A_{rk} \cos (n_k t + \phi_k), \tag{86}$$

in which, it will be remembered, B_k and ϕ_k are arbitrary, independent of r, while A_{rk} must be consistent with the ratios $\left(\frac{\xi_r}{\xi_1}\right)_{n=n_k}$ as given by the appropriate equations from the set (78). Now in (78) we always have

$$-\xi_{r-1} + C\xi_r - \xi_{r+1} = 0,$$

or, for $n = n_k$ from (86)

$$-A_{r-1,k} + C_k A_{rk} - A_{r+1,k} = 0,$$

or

$$C_k A_{rk} = A_{r+1,k} + A_{r-1,k};$$

and if $C_k = 2 \cos \theta_k$, since

$$2 \cos \theta_k \sin r\theta_k = \sin (r+1)\theta_k + \sin (r-1)\theta_k,$$

we may take

$$A_{rk} = \sin r\theta_k. \tag{87}$$

Now $\theta_k = \frac{k\pi}{m+1}$, as previously determined (cf. eq. 83), and therefore

$$A_{rk} = \sin \frac{rk\pi}{m+1}. \tag{87a}$$

We have finally, for the complete solution

$$\xi_r = \sum_{k=1}^{m} B_k \sin \frac{rk\pi}{m+1} \cos (n_k t + \phi_k), \tag{86a}$$

with the $2m$ constants B_k and ϕ_k subject to initial conditions.

In view of the connection between steady state and transient phenomena in structures of this sort, which we have previ-

ously emphasized, it should not be necessary to go further to obtain an idea of the response of the system when steadily driven by force of frequency $\frac{\omega}{2\pi}$; but on account of the importance of the problem we desire to outline the behavior of the system when ω is varied, and so study its filtering properties.

Let a force $\Psi_1 = \Psi_0 e^{i\omega t}$ be applied to particle (1). Then we have for the displacements of the particles nearest the beginning and the end of the string (cf. eq. 60).

$$\xi_1 = \frac{M_1}{D_m}\Psi_1 \quad \text{and} \quad \xi_m = \frac{M_m}{D_m}\Psi_1, \tag{88}$$

and since the minors $M_1 = D_{m-1}$, and $M_m = 1$ we have

$$\frac{\xi_m}{\xi_1} = \frac{1}{D_{m-1}}. \tag{88a}$$

Now we know, that if m is large, D_{m-1} has a succession of $(m - 1)$ roots within the same region of frequency as those of D_m, that is, within the region of natural oscillations, and that in this frequency range $\frac{\xi_m}{\xi_1}$ has appreciably large values, which is to say that oscillations produced at one end of the structure travel easily to the other end. The question is, what is the value of the ratio $\frac{\xi_m}{\xi_1}$ when the driving force has a frequency slightly in excess of the highest natural frequency of the system, and this depends on the evaluation of D_{m-1} for $\omega > 2\pi f_0$.

In the expression $C = 2 + \frac{\lambda^2 a_1}{c_1}$ we substitute $-\omega^2$ for λ^2, and $\frac{1}{4}n_0^2$ for $\frac{c_1}{a_1}$ (cf. eq. 85b), thus

$$C = 2\cos\theta = 2 - 4\frac{\omega^2}{n_0^2} \quad \text{or} \quad \cos\theta = 1 - 2\frac{\omega^2}{n_0^2}; \tag{89}$$

and it is seen that if $\omega > n_0$, θ becomes imaginary; but we must

determine θ, to determine D_{m-1}. Now if $\theta = ix + \pi$, we may write $-\cosh x = \cos(ix + \pi)$, and

$$\left.\begin{aligned} D_{m-1}(C) &= \frac{\sin m(ix+\pi)}{\sin(ix+\pi)} = \frac{(-1)^{m-1}\sinh mx}{\sinh x}, \\ \text{in which} \qquad & \\ \frac{C}{2} &= 1 - 2\frac{\omega^2}{n_0^2} = -\cosh x. \end{aligned}\right\} \tag{90}$$

The general course of $D_{m-1}(C)$ as a function of $\frac{\omega}{n_0}$ is indicated

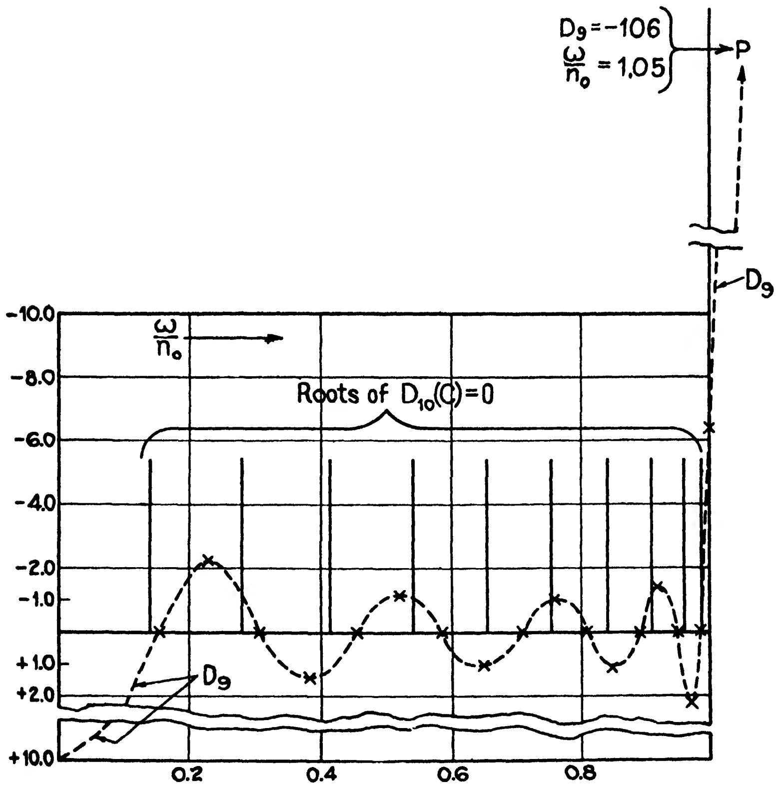


FIG. 9.—PROPERTIES OF $D_{10}(C)$ AND $D_9(C)$ (LOADED STRING).

in Fig. 9, for the case $m = 10$. The roots of $D_{10}(C)$, (the 10 natural frequencies of the string) are also indicated by the verti-

cal lines. A study of this particular case, based on the behavior of $D_{10}(C)$ and $D_9(C)$ shows clearly the filtering action of a string equally loaded at only 10 points. At the point P, which represents D_9 for a value $\frac{\omega}{n_0} = 1.05$, computed by means of (88) the relatively small value of $\frac{\xi_{10}}{\xi_1} = \frac{1}{D_9}$ is strikingly shown, as compared with values of $\frac{1}{D_9}$ in the resonant range. As $\frac{\omega}{n_0}$ is further increased, D_9 still rises rapidly and it is evident that the string acts as a very efficient low-pass filter, cutting off all vibrations of frequency greater than $\frac{n_0}{2\pi}$. If damping had been taken into account, the rise in D_9 outside the transmission range would have been less rapid, but still sufficient to show the filtering effect.

The general consideration of filters as such would take us far afield. But we must not overlook the practical advantage to be gained, if in any mechanical case we are able to substitute the corresponding electrical network; for there is available a complete theory of electrical structures of this kind, and their properties are very well understood.[1] The electrical theory in its simplest form relates to the steady state behavior of an

[1] Care must be taken to preserve dimensional relations in comparing mechanical systems to their electrical analogues. In most parts of the text it is convenient to take the *mechanical* impedance as the ratio of maximum force to maximum velocity. Strictly, the velocity is analogous to *current density* rather than to *total current,* and in dealing with filters or similar structures this should be taken into account. It is evident that we must consider the flux $S\dot{\xi} \equiv \dot{X}$ as the equivalent of the electric current I. For example, the kinetic energy in an orifice is:

$$T = \frac{1}{2}\frac{\rho S^2 \dot{\xi}^2}{K} = \frac{1}{2}\frac{\rho}{K}\dot{X}^2.$$

This is analogous to the kinetic energy in an inductance, $\frac{1}{2}LI^2$; the inductance L is then analogous to the quantity $\frac{\rho}{K}$. The reader can easily make the necessary applications of this idea, to the stiffness and resistance factors which also enter into the impedance. To look at the matter in another way, the acoustic impedance *per unit area, divided by the area,* is what corresponds most closely to the electrical impedance.

iterated structure of the ladder type, the longitudinal members of which are known as series impedances (Z_1) while the transverse members (the rungs of the ladder) are known as shunt impedances (Z_2). The first important property of such a network is its iterative impedance, that is, its driving-point impedance when the structure is infinitely long. It is easy to show that the iterative impedance (at mid series) is

$$Z_k = \sqrt{Z_1 Z_2}\sqrt{1 + \frac{Z_1}{4Z_2}}. \tag{90a}$$

Now, defining the propagation constant[1] P as the logarithm of the ratio of the current in one section to that in the preceding section, that is,

$$\frac{I_{(n+1)}}{I_n} = e^P, \tag{90b}$$

it follows that

$$\left.\begin{aligned} P &= 2 \log \left(\frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} + \sqrt{1 + \frac{Z_1}{4Z_2}}\right) \\ &= \alpha + i\beta. \end{aligned}\right\} \tag{90c}$$

The factors α and β are quite analogous to the attenuation and phase factors which we shall encounter in Chap. III, in dealing with wave propagation in acoustic media. Equation (90c) may be written

$$P = 2 \sinh^{-1} \frac{1}{2}\sqrt{\frac{Z_1}{Z_2}} = 2i \sin^{-1} \frac{1}{2}\sqrt{-\frac{Z_1}{Z_2}}, \tag{90d}$$

from which it follows that if the value of $\frac{Z_1}{Z_2}$ falls within the range $-4 < \frac{Z_1}{Z_2} < 0$, P can then be expressed as a pure imaginary ($P = i\beta$) and $\alpha = 0$; that is to say, there is *no attenuation within the corresponding frequency range* and the limiting frequencies

[1] This term is a misnomer, but its usage is firmly fixed in the language. P is not a constant; it is a function of the impedance, and hence of the frequency.

so defined are the boundaries of a frequency transmission band of the filter. The conditions implied in this statement are realized strictly only if we neglect dissipation; if dissipation is taken into account, there will, of course, be attenuation in the transmission band; but for moderate values of dissipation the limiting frequencies determined from the condition $-4 < \frac{Z_1}{Z_2} < 0$ by considering reactances only, will not be appreciably in error.[1] The elements of the simple theory of wave filters which we have outlined will find application in the next section and in problems 17 and 18 at the end of the chapter.

27. *Acoustic Filters*

Following the development of electrical filters, G. W. Stewart devised in 1920 (*Phys. Rev.*, March, 1921, p. 382) an acoustic low-pass filter, the theory of which is in all respects parallel to that of the low-pass electrical filter, or of the string of beads which we have just considered. The acoustic filters are merely iterated resonators, and presumably for every electrical filter known there is an acoustic analogue, though all are not equally practicable.

In Figs. 10*a*, 10*b*, 10*c* are shown respectively diagrams of low-pass mechanical, acoustic and electrical filter structures, the arrows indicating the displacement or velocity of motion for the normal *mode of highest frequency*, that is, the limiting frequency of transmission. In each case the filter is supposed to have an infinite number of successive equal elements; the oscillations are exactly out of phase from one element to the next, in the mode considered. The simple formulae for limiting frequency, assuming inertia and stiffness entirely concentrated at a point, and taking no account of resistance, are shown adjacent to the diagrams.

[1] For a more extensive account of filter theory, the reader may refer to K. S. Johnson, "Circuits for Telephonic Communication," Chaps. XI and XVI. Many papers by Bell System Engineers on Electrical Filters have appeared in the *Bell System Technical Journal* e.g., O. J. Zobel, *ibid.*, Jan., 1923, p. 1.

If the conductivity k_2 of the side orifices into the volume V_2 is taken into account, the more accurate formula is

$$f_0 = \frac{c}{\pi}\sqrt{\frac{S_1}{l_1 V_2}\left[\frac{1}{1+\frac{4S_1}{k_2 l_1}}\right]} = \frac{c}{\pi}\sqrt{\frac{k_1}{V_2}\left[\frac{1}{1+\frac{4k_1}{k_2}}\right]}, \qquad (B')$$

according to a later paper of Stewart's which deals with the general theory of Acoustic Wave Filters (*Phys. Rev.*, Dec.,

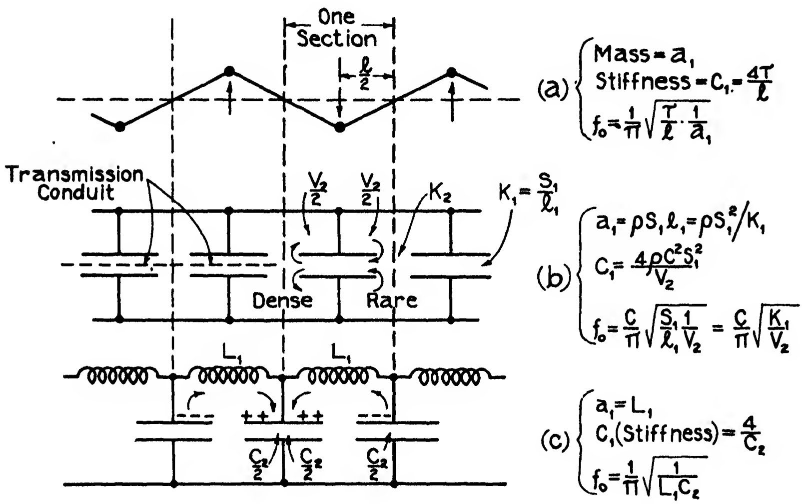


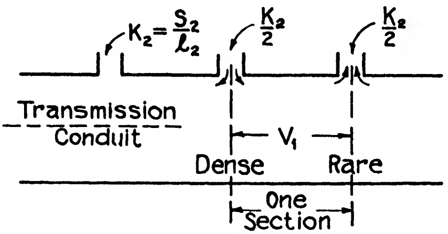
FIG. 10.—LOW-PASS FILTER ANALOGUES.

1922, p. 528). This may be obtained by looking upon the conductivity k_1 as requiring a correction, due to the conductivities each $\frac{k_2}{2}$ connected in series at either end of k_1; thus if k'_1 is the corrected value we should have, adding "resistances,"

$$\frac{1}{k'_1} = \frac{1}{k_1} + \frac{2}{k_2} + \frac{2}{k_2}, \quad \text{or} \quad k'_1 = \frac{k_1}{\left(1+\frac{4k_1}{k_2}\right)},$$

which accounts for the difference between the formulae in Fig. (10*b*) and equation (B'). The latter equation "gives approxi-

mately the experimental values of f_0 and also explains satisfactorily the variation in f_0 with the conductivity of the orifices leading from the transmission conduit to the volume in the branch." (Stewart.) Note that in Stewart's formulae subscripts (1) relate to properties of the "transmission conduit" while subscripts (2) relate to branch elements which are in shunt with the transmission conduit.



$$\begin{cases} f_0 = \frac{C}{2\pi}\sqrt{\frac{K}{V}} \\ V = V_1 ; \frac{1}{K} = \frac{2}{K_2} + \frac{2}{K_2} \\ \therefore f_0 = \frac{C}{2\pi}\sqrt{\frac{K_2}{4V_1}} \end{cases}$$

FIG. 11.—ACOUSTIC HIGH-PASS FILTER (SCHEMATIC).

The high-pass filter is shown schematically in Fig. 11. The approximate formula for f_0 is shown adjacent to the figure. The more accurate formula preferred by Stewart is

$$f_0 = \frac{c}{2\pi}\sqrt{\frac{k_2}{4V_1}\left(1 + \frac{4k_1}{k_2}\right)}. \qquad (E')$$

This is directly obtainable from the formula of Fig. 11, if we consider that the conductivity $2k_1 \equiv \frac{2S_1}{l_1}$ of a section of the transmission conduit is disposed as a shunt with respect to the half conductivity $\frac{k_2}{2}$ of each of the orifices which serve a given section.

No ready explanation is forthcoming for the failure of the published attenuation curves (Stewart, *loc. cit.*, and later: *Phys. Rev.*, April, 1924, p. 520; also Peacock in the paper immediately following) to compare favorably with those of the analogous electrical filters in everyday use. (Note that Stewart's curves show relative transmission to a linear scale, and not to a loga-

rithmic scale, in *transmission units*,[1] as is customary in the telephone laboratory.) The damping $\frac{R}{2L}$ characteristic of one section of a good electrical filter can be made small, e.g., 10 or less; but so indeed can the damping of the portion of the acoustic transmission conduit which serves as the neck of the corresponding resonator element. (For example, applying (69*e*), $\Delta_2 = \frac{1}{r}\sqrt{\frac{\mu\omega}{2\rho}}$, substituting $r = .75$, which applies roughly to the filter structures of Stewart, we have $\Delta_2 = 24$, taking $\frac{\mu}{\rho} = .13$ and $\omega = 5 \times 10^3$.) Stewart has very properly made the main transmission conduit a relatively wide tube; but the side conductivities (k_2) are likely to introduce complications particularly as regards dissipation. The side conductivities in the low-pass filters are due to a number of small holes; and we should expect considerable dissipation therein. In the high-pass filter the side conductivities will cause additional losses due to radiation. Allowance for these complications and for the fact that the inertia and stiffness elements are much more definitely "lumped" in the electrical case may explain the *relative* shortcomings of the acoustic filter.

A disadvantage due to multiple resonance[2] in the resonators (which accounts for the failure of the low-pass filter to suppress very high frequencies) is apparently inherent, due to the continuous structure of the acoustic medium within the resonator. This disadvantage of course is lacking in discretely loaded structures, such as the string of beads. It is an experimental

[1] Two power levels, W_1, W_2, are said to differ by one transmission unit (TU) when they are in the ratio $\frac{W_1}{W_2} = 1.25$ (approximately). The general formula is

$$N_{TU} = 10 \log_{10}\left(\frac{W_1}{W_2}\right).$$

See K. S. Johnson, "Transmission Circuits," p. 12.

[2] See the curves in Peacock's paper, referred to above; also a later paper by Stewart (*Phys. Rev.*, 25, p. 90, Jan., 1925) in which he gives the necessary extension of the theory.

fact that, by means of springs and weights very selective structures can be built, designing them for transverse, longitudinal or torsional vibrations.

28. *Finite String; Normal Coordinates; Normal Functions*

We conclude with a discussion of the motion of a finite stretched string of uniform density. There is a certain interest in the problem as such, and in addition, we can use it to introduce a further consideration of more general matters.

According to the straightforward method (cf. § 10) take ρdx for the mass of an element of the string, the displacement being $\xi(x)$; then since the net force in the direction of ξ due to the tension τ is $\tau\frac{\partial}{\partial x}\left(\frac{\partial\xi}{\partial x}\right)dx$, the equation of motion is,

$$\rho\ddot{\xi} = \tau\frac{\partial^2\xi}{\partial x^2} \qquad \text{or} \quad \ddot{\xi} = c^2\frac{\partial^2\xi}{\partial x^2} \qquad \text{if} \quad c^2 = \frac{\tau}{\rho}. \tag{91}$$

Without considering in detail the substitution $\xi = \phi(x)e^{\lambda t}$ it is evident that one solution is of the type

$$\xi = A\cos(nt - \theta)\sin\frac{nx}{c}, \tag{92}$$

A and θ being arbitrary while n is to be related to the dimension l and the constant c of the system. For a string of length l, choosing the origin at one end of the string, we must have $\xi = 0$ when $x = 0$ and $x = l$, whence

$$\frac{nl}{c} = k\pi \qquad (k \text{ any integer})$$

and the natural frequencies are

$$n_1 = \frac{\pi c}{l}, \quad n_2 = \frac{2\pi c}{l}, \quad \ldots \quad n_k = \frac{k\pi c}{l}; \quad \left(c^2 = \frac{\tau}{\rho}\right). \tag{93}$$

Thus, for any of the k normal modes

$$\xi = A_k\cos(n_k t + \theta_k)\sin\frac{k\pi x}{l}, \tag{94}$$

or, for any mode of arbitrary initial form, superposing in any desired proportion a series of solutions of the type of (94),

$$\xi(x) = \sum_{k=1}^{m} A_k \cos (n_k t + \theta_k) \sin \frac{k\pi x}{l}. \tag{94a}$$

We may now compare the result as to n_k with that which we should obtain indirectly from the problem of the loaded string, letting the spacing become so minute that the effect is of "continuous" loading. Rewriting equation (85) and noting that the spacing l of the former problem is the equivalent of dx in the present case, we have, since

$$c_1 = \frac{\tau}{dx}, \qquad a_1 = \rho dx, \qquad (m + 1)dx = l,$$

$$n_k = 2\sqrt{\frac{c_1}{a_1}} \sin \frac{k}{m+1}\frac{\pi}{2} = 2\sqrt{\frac{\tau}{\rho dx^2}} \sin \frac{kdx}{l}\frac{\pi}{2};$$

or, since the argument of the sine function is infinitesimal, and we can replace the sine by the argument itself,

$$n_k = \frac{k\pi}{l}\sqrt{\frac{\tau}{\rho}} = \frac{k\pi}{l}c.$$

Now, as to the factor $\sin \frac{k\pi x}{l}$, which must be obtained from (87*a*), viz.:

$$A_{rk} = \sin \frac{rk\pi}{m+1}, \tag{87a}$$

we note that for the rth element dx, as we proceed along the string, $rdx = x$, and since $(m + 1)dx = l$,

$$A_{rk} = \sin \frac{x}{dx}\frac{dx}{l}k\pi = \sin \frac{k\pi x}{l}. \tag{95}$$

To summarize, proceeding from the string loaded with discrete

particles, to the continuously loaded string, we have in the limit

$$\xi_r = \sum_{k=1}^{m} B_k A_{rk} \cos(n_k t + \phi_k) \doteq \xi(x) = \sum_{k=1}^{m} B_k \cos(n_k t + \phi_k) \sin\frac{k\pi x}{l}, \tag{95a}$$

a result identical with (94*a*). This was the method followed by Lagrange, to whom we owe the solution of the beaded string problem. The reader interested in the rapidity of convergence for the fundamental mode will find an interesting table in Rayleigh, Vol. I, §120. It appears, for example, that for the case of equal discrete loading at as many as six points, the frequency is within about 1 per cent of that of the continuously loaded string of the same total mass, length and tension. We shall not carry the matter further.

One of the interesting questions which we wish to consider is the relation of such dynamical developments as that of (94*a*) to the more purely mathematical notions of the properties of Fourier's Series. To obtain (94*a*), we have (consciously) used [apart from the fact that sines and cosines are standard solutions of (91)] only one mathematical idea, viz.: the principle of superposition. This is that any finite sum of solutions of a linear differential equation is also a solution. If this is granted, and if the summation is extended to include a very large number of separate solutions, then in solving the problem of a uniform, continuous string we have virtually a dynamical proof of the validity of the expansion of the arbitrary initial configuration of the string into a sine or cosine series. From the practical standpoint it is reassuring to note that "the physical induction has been most fully corroborated by independent mathematical proof." (Lamb, § 39, which see by all means.) The reader will enjoy discovering for himself, if he has not already done so, the parallelism all along the way between the analytical theory of Fourier's series and the purely physical ideas involved in the application of these series to concrete problems. Instead of the well known analytical device of integration, for example, to

determine the coefficients A_k in (94a), we can determine them to as high a degree of approximation as we may desire by taking a sufficient number of points on the string, writing for each (for $n_k t + \theta_k = 0$),

$$\left.\begin{aligned} \xi(x_1) &= A_1 \sin\frac{\pi x_1}{l} + A_2 \sin\frac{2\pi x_1}{l} + \dots \\ \xi(x_2) &= A_1 \sin\frac{\pi x_2}{l} + A_2 \sin\frac{2\pi x_2}{l} + \dots \\ &\dots\dots\dots\dots \end{aligned}\right\} \tag{96}$$

and solving the set of simultaneous equations. This is sound enough from the physical standpoint, but generally more cumbersome than the integration method according to which

$$A_k = \frac{2}{l}\int_0^l \xi(x)\sin\frac{k\pi x}{l}dx, \tag{96a}$$

an easily proved mathematical relation. To follow the ideas we have in mind, in making these brief suggestions, the chapter on Fourier's Theorem in Lamb will be found very much to the point.

Again consider the matter of normal coordinates. In dealing with the *continuously* loaded string we have arrived, as it happens, at a solution in terms of normal coordinates; we began, with the string loaded with discrete particles, using coordinates that were not normal. In §10 (p. 25) we anticipated matters to a certain degree by stating the fact that motions due to arbitrary symmetrical distensions of the circular membrane could be described by means of a certain series, each term of which implies a normal coordinate. (The difference between the factors $\sin\frac{k\pi x}{l}$ and $J_0(k_j r)$ in the two cases is not material at the moment.) It seems then, that for the string and the membrane (in both of which the mass is uniformly distributed) the use of normal coordinates (and the Principle of Superposition) in attacking the problem is a natural procedure; whereas in the case of non-uniform distributions of mass, it would be

impracticable. In general, while we may *not always* be successful in applying the method of normal coordinates to the vibrations of a system with uniformly distributed constants, it is to such systems that we look for the principal applications of the method.

Finally, the success of the method of normal coordinates, and the resulting gain of considering each natural oscillation as independent of the others, depends on the availability of a *set of related normal functions* in terms of which the *shape* of the vibrating body can be stated at any moment of its history. Mathematically a set of functions $F_1(x)$, $F_2(x) \ldots F_n(x)$ is *normal in the interval* (a, b) if

$$A\int_a^b F_k^2(x)dx = 1, \tag{97}$$

in which $\sqrt{A}$ is known as a *normalizing constant*. The set of functions, for example,

$$\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \ldots \frac{\sin kx}{\sqrt{\pi}}; \quad \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots \frac{\cos kx}{\sqrt{\pi}};$$

is normal in the interval $(0, 2\pi)$. A set of Bessel's functions can likewise be normalized.

Now it is a fact that the *normal* functions which are useful in physical problems are likewise *orthogonal*: they satisfy the relation

$$\int_a^b F_j(x)\cdot F_k(x)dx = 0 \quad \text{for} \quad j \neq k, \tag{98}$$

the interval (a, b) being taken as before. Thus we have

$$\left.\begin{aligned} &\int_0^{2\pi} \sin jx \sin kxdx = 0, \text{ etc.,} \quad j \neq k; \\ &\int_0^a xJ_n(\alpha_j x)J_n(\alpha_k x)dx = 0, \quad J_n(\alpha_j a) = J_n(\alpha_k a) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \neq k; \end{aligned}\right\} \tag{98a}$$

to give only two examples. The latter set is of special interest

in connection with the problem of the circular membrane (§10, p. 25); we stated that any symmetrical initial distension could be given by a series of the form (simplifying (29) by letting $\theta_1 = \theta_2 = 0$),

$$\xi(r) = A_1 J_0(\alpha_1 r) + A_2 J_0(\alpha_2 r) + \ldots, \tag{29a}$$

and this is true provided the coefficients A_k are given by

$$A_k = \frac{2}{a^2[J_1(a_k r)]^2}\int_0^a r \cdot \xi(r) \cdot J_0(\alpha_k r)dr, \tag{99}$$

in which a is the radius of the membrane, and $J_0(\alpha_k a) = 0$.

We state these results briefly, and without proof, in order to give the reader an idea of the general situation in regard to the use of normal functions in solving the problem of normal modes of oscillation in certain systems. Fortunately, the appropriate normal functions are known for the uniform string, and the uniform membrane. (In the particular case of the string, and similar problems for which the *normal functions are trigonometric functions*, the expansion of a given function into series of sines and cosines is known as the method of Fourier.) In the case of vibrations of a *rectangular plate, the normal functions are unknown*; but in most problems with which we have to deal, the properties of a certain set of known normal functions are the key to the solution. Following the chapter on Fourier's Theorem in Lamb, the more general treatment in Whittaker and Watson's "Modern Analysis" will provide the necessary background in the theory of normal functions.

PROBLEMS

11. Solve the problem of § 21—[i.e., obtain equations (55a)] using two normal coordinates.

12. An electrical circuit (telephone receiver winding) of inductance L and resistance R is arranged so that it exerts on a vibrating member (diaphragm) a force Q_1 (dynes) for each c.g.s. unit of current

flowing. The moving member induces in the circuit an electromotive force Q_2 (c.g.s.) per unit of velocity. Taking the constants of the vibrating member as a, b, c (in the familiar notation), find the driving point impedance of the electrical circuit when the vibrating member is in resonance.

13. (*a*) A spherical resonator with thin walls contains 400 cu. cm., and has a circular orifice of radius 1 cm. Find the natural frequency and the damping coefficient, taking the velocity of sound in air at ordinary temperatures as 3.4×10^4 cm./sec.

(*b*) The orifice of the resonator is now fitted with a tube of diameter 2 cm. and length 4 cm. Compute the natural frequency, and the damping due to radiation under these new conditions; also give an estimate of the damping due to friction in the neck.

14. In § 25 a certain problem is solved for free oscillations. From data there given, write the equations for the steady state velocities $\dot{\xi}_1$ and $\dot{\xi}_2$ when a force $\Psi_0 \cos \omega t$ is applied at the center of the diaphragm, and compute the values of the ratio $\frac{\dot{\xi}_2}{\dot{\xi}_1}$ at the resonant frequencies of the system.

15. A string of 3 equally spaced beads is initially displaced so that $\xi_1 = +1$, $\xi_2 = 0$ and $\xi_3 = -1$. Taking $\frac{c_1}{a_1} = 10^7$ determine the constants B_k, A_{rk}, n_k, and ϕ_k of equation (86) and (86*a*) for zero initial velocity.

16. A flexible uniform string of length l is displaced and let go, the displacements of points distant $\frac{l}{4}$, $\frac{l}{2}$ and $\frac{3l}{4}$ from one end being respectively $+1$, 0, -1. Adjusting the fundamental of this string to agree with that of problem (15), compare the motion of the three corresponding points spaced along the 2 strings. (Use the first four terms which do not vanish of the Fourier's series corresponding to (94*a*).)

17. For the loaded string (§ 26) construct the equivalent electrical filter (that is, determine the mechanical reactances corresponding to L and C), and verify this by finding the relation between the quantity $\frac{Z_1}{Z_2}$ and the quantity $C = 2 \cos \theta$, (eq. 82) at the limiting frequencies of transmission.

18. For the filter sketched in Fig. 12, sketch the electrical analogue, and on this basis or by any other available method determine the limiting frequencies of the filter.

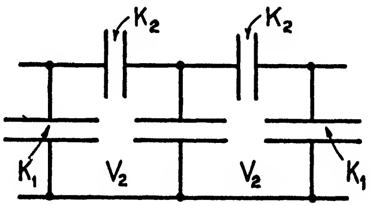


FIG. 12.—SEE PROBLEM 18.

19. Devise a low-pass torsional filter suitable for coupling between two rotating shafts, and determine its upper limiting frequency.

20. A double resonator consists of two coupled resonators each of volume V_0. The first resonator communicates to the air through an orifice of area S and conductivity K; the connection between the two resonators is through a like opening. A plane wave of pressure amplitude $p_1e^{i\omega t}$ is incident on the first orifice. Neglecting damping in the inner orifice, find the natural frequencies of the coupled system, and the overall velocity amplification of the system at these frequencies. How do these coefficients compare with the velocity amplification of the first resonator alone, at its resonant frequency? How would you diminish the coupling between the resonators, and what would be the effect of this on the amplification at the selected frequency? (E. T. Paris, *Science Progress*, XX, No. 77, 1925, p. 68.)

CHAPTER III

The Propagation of Sound

30. *Properties of the Medium; Equation of Wave Motion*

In Chapter I we dealt with the simplest type of vibrating system. According to the point of view there adopted, all parts of the system moved in phase, that is, any particular state of vibration was manifested simultaneously in all parts of the system. In Chapter II we have seen the result of coupling together a number of equal simple systems in a long structure; we have observed that the vibrations of such a structure may be regarded as wave phenomena, since there are periodic changes of phase in passing from each element of the structure to the next; and we have noted that in the limit, the wave-transmitting properties of a long structure made up of equal discrete particles, each equally coupled to the next, tend to approximate those of a continuous medium. It is both logical and expedient, at this point, to take up the subject of wave propagation in acoustic media.

Sound waves are the inevitable result when vibratory stresses are applied at the boundary, or in any part of a compressible fluid. The physical properties of the medium must therefore be stated; and we consider first the *specific* properties in terms of the mean volume and density at a particular point in the fluid. These are, for small increments of volume and density,

(*a*) Dilatation

$$\Delta = \frac{\delta v}{v_0}, \quad \text{i.e.,} \ v = v_0(1 + \Delta); \tag{101}$$

(*b*) Condensation,

$$s = \frac{\delta \rho}{\rho_0}, \qquad \rho = \rho_0(1 + s). \tag{102}$$

(From (*a*) and (*b*) note that since $\rho v = \rho_0 v_0$,

$$s = -\Delta \quad \text{and} \quad (1 + s)(1 + \Delta) = 1, \tag{103}$$

neglecting the product $s\Delta$ as a small quantity of the second order.)

For a fluid perfectly elastic as to small condensations we have, experimentally,

$$-\frac{\delta v}{v_0} \cdot \frac{1}{\delta p} = C, \qquad C = \frac{1}{\kappa}, \tag{104}$$

in which C is the compressibility, and κ the bulk modulus, or coefficient of cubic elasticity. If the fluid is a gas undergoing slow (i.e., isothermal) changes in volume, we also have $pv = p_0 v_0$ whence

$$\frac{\delta p}{p_0} = -\frac{\delta v}{v_0} = s, \tag{104a}$$

and the *coefficient of cubic elasticity* is

$$\left.\begin{aligned} &(c) && \kappa = \frac{\delta p}{-\dfrac{\delta v}{v_0}} \equiv p_0 \equiv \frac{\delta p}{s}, \\ &\text{whence} && \\ & && p = p_0 + \kappa s. \end{aligned}\right\} \tag{105}$$

For a perfect gas we have relations (101–103) as above, but, for rapid changes in p and v, we must use the adiabatic relation $pv^\gamma = p_0 v_0^\gamma$ hence in this case

$$\frac{\delta p}{p_0} = -\gamma\frac{\delta v}{v_0} = \gamma s \tag{104b}$$

and the *coefficient of cubic elasticity* is

$$\left.\begin{aligned} &(c') && \kappa' = \gamma p_0 = \frac{\delta p}{s}, \\ &\text{whence} && \\ & && p = p_0 + \kappa' s. \end{aligned}\right\} \tag{105a}$$

The *general* relation which must hold for a given small volume on any fluid, is the following:

"The difference between the amounts of fluid which flow in and out of a small closed surface during a small interval of time δt must be equal to the increase in the amount of fluid during the same interval, which the surface contains." This is the *principle of continuity*; we proceed to give it analytical form, for motion in one dimension, namely along the x axis. Let the volume considered be a lamellar element with opposite faces of area A, normal to the x axis, the thickness of the element being dx. Then the net flow (or *flux*) through the faces of the element in time δt is

$$\delta t\left[\rho\dot{\xi} - \left(\rho\dot{\xi} + \frac{\partial}{\partial x}(\rho\dot{\xi})dx\right)\right]A = -Adx\cdot\frac{\partial}{\partial x}(\rho\dot{\xi})\,\delta t,$$

while the increase in the amount of fluid contained is

$$\delta t\cdot Adx\cdot\frac{\partial\rho}{\partial t};$$

hence the relation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho\dot{\xi}) = 0, \tag{106}$$

which is the *equation of continuity*. This kinematical relation, which can easily be generalized for motion in three dimensions, is probably the most useful one in hydrodynamics. Replacing ρ by $\rho_0(1 + s)$ in (106) and neglecting $s\dot{\xi}$ as compared with $\dot{\xi}$, we have as more suitable for our purpose

$$\frac{\partial s}{\partial t} + \frac{\partial\dot{\xi}}{\partial x} = 0, \tag{106a}$$

whence

$$\frac{\partial^2 s}{\partial t\partial x} + \frac{\partial^2\dot{\xi}}{\partial x^2} = 0. \tag{106b}$$

Now if p is the excess pressure on one face of a lamellar element $A'dx$, and $p + dx\cdot\frac{\partial p}{\partial x}$ the pressure on the opposite face, then the net force due to pressure is $-A'dx\cdot\frac{\partial p}{\partial x}$, and taking into ac-

count the rate of change of momentum of the element we have the equation of motion

$$A'dx\left(\frac{\partial p}{\partial x} + \rho\ddot{\xi}\right) = 0. \tag{107}$$

Writing for p, κs (105), and differentiating (107) with respect to time, we have

$$\kappa\frac{\partial^2 s}{\partial x dt} + \rho\frac{\partial^2 \dot{\xi}}{\partial t^2} = 0, \tag{107a}$$

and by comparison of (107*a*) and (106*b*)

$$\frac{\partial^2 \dot{\xi}}{\partial t^2} = \frac{\kappa}{\rho}\frac{\partial^2 \dot{\xi}}{\partial x^2} \equiv c^2\frac{\partial^2 \dot{\xi}}{\partial x^2}. \tag{108}$$

This is the equation for the propagation of a *sound wave*, (*compressional or longitudinal wave*) in which $c = \sqrt{\frac{\kappa}{\rho}}$ is the velocity of propagation. $\dot{\xi}$ is the "particle velocity" at any point in the medium; the same equation would be obtained for ξ as for $\dot{\xi}$. The same equation (as regards form) was obtained in studying the problem of the vibrating string (*cf.* 91) but the wave velocity in that case $\left(\sqrt{\frac{T}{\rho}}\right)$ related to the propagation of *transverse* disturbances along the string. In perfect fluids, only longitudinal waves need be considered; in solids both longitudinal and transverse waves are possible, the latter case depending on the elasticity of the medium to shearing stresses. All the steps leading to equation (108) apply equally well to the case of an elastic solid rod, provided that for κ the longitudinal elastic constant (Young's Modulus) is used. The physical constants of various solid and fluid media, including $c = \sqrt{\frac{\kappa}{\rho}}$ and $R = \sqrt{\kappa \cdot \rho}$, the two principal constants relating to sound transmission, are given in Drysdale, "Mechanical Properties, etc.," pp. 288–292.

31. *Properties of Plane Waves of Sound*

We propose to examine the more important properties of plane (aerial) waves of sound, starting from the fact that each of the quantities (ξ, $\dot{\xi}$, ρ, s, Δ, δp) which relate to the state of an element of the medium must equally satisfy the equation:

$$\frac{\partial^2\theta}{\partial t^2} = c^2\frac{\partial^2\theta}{\partial x^2}, \qquad \theta = \xi,\ \dot{\xi},\ \Delta,\ s,\ \rho \text{ or } \delta p. \tag{108}$$

From this equation we infer only the fact that solutions must be of the type

$$\theta = A\cdot f(ct - x) + B\cdot F(ct + x), \tag{109}$$

that is to say, the general solution implies for any of the factors θ, *two wave trains*, each of *arbitrary* form, and proceeding in opposite directions with velocity c. The values of A and B, and the particular forms of $f(ct - x)$ and $F(ct + x)$ must be determined by the assigned boundary values in a particular problem. The choice of which factor θ (e.g., condensation, excess pressure, particle velocity or what not) we wish to use in a given case depends on the nature of the problem.

Thus in a closed tube, with recurrent reflections at either end, we might, as in the string problem, wish to study the displacement ξ; the solution would then be in the form

$$\xi = A\left[\sin\left(\omega t + \frac{k\pi x}{l}\right) - \sin\left(\omega t - \frac{k\pi x}{l}\right)\right] = 2A\cos\omega t\sin\frac{k\pi x}{l}, \tag{109a}$$

which clearly satisfies the differential equation and the boundary conditions if

$$\frac{\omega l}{k\pi} = c, \quad \text{that is if } \lambda_k = \frac{2l}{k}, \quad \text{since } f_k\cdot\lambda_k = c \tag{109b}$$

λ being the wave length, and f_k the frequency of one of the natural oscillations of the tube. Suppose, however, we study the

propagation of a single pressure impulse whose original wave form is given by the expression

$$\left.\begin{aligned} \delta p|_{t=0} = f(x) &= 1, \quad \text{for } 0 < x < x_0 \\ &= 0 \text{ elsewhere;} \end{aligned}\right\} \qquad (109c)$$

then, (neglecting dissipation due to friction) and considering propagation in only the positive direction, we must have

$$\left.\begin{aligned} \delta p = f(x - ct) &= 1, \quad \text{for } ct < x < ct + x_0 \\ &= 0 \text{ elsewhere.} \end{aligned}\right\} \qquad (109d)$$

These instances emphasize the purely kinematical information which is available from (108).

Now consider a dynamical question, that of energy in a plane wave. A piston of unit area is given an oscillating motion $\xi_0 \cos \omega t$, and this motion is communicated to the near end of a column of air of *unit cross section*, which extends to infinity. At any distance x along the column we must have, for the particle displacement

$$\xi(x) = \xi_0 \cos \omega \left(t - \frac{x}{c}\right), \qquad (110)$$

consistently with (108), (109) and the special conditions we have named.

Consider the *potential energy* first. The excess pressure δp producing any given condensation s is $\kappa' s$. (*Cf.* 105*a*). Thus in any lamellar element of volume dx we have

$$dx \cdot V = dx \int_0^s \kappa' s \, ds = \tfrac{1}{2} dx \cdot \kappa' s^2. \qquad (111)$$

The kinetic energy is

$$dx \cdot T = \tfrac{1}{2} dx \cdot \rho \dot{\xi}^2. \qquad (112)$$

We can easily show that the *kinetic energy is equal to the poten-*

tial energy at any point in a plane wave train. If ξ is the displacement at point x, then at $x + dx$ the displacement is

$$\xi + \left(\frac{\partial \xi}{\partial x}\right) dx,$$

and we have

$$s \equiv -\Delta = -\frac{\partial \xi}{\partial x}. \tag{113}$$

Now from (110)

$$\dot{\xi} \equiv \frac{d\xi}{dt} = -c\frac{\partial \xi}{\partial x}. \tag{114}$$

hence

$$cs = \dot{\xi} \quad \text{and} \quad dx \cdot T = \tfrac{1}{2} dx \cdot \rho c^2 s^2 = \tfrac{1}{2} dx \cdot \kappa' s^2, \tag{112a}$$

a result [1] identical with (111). Thus the total energy $dx(T + V)$ is half Kinetic and half Potential and "since $\dot{\xi}$ has the same sign as s an air particle moves forward (i.e., with the waves) as a phase of condensation passes it, and backwards during a rarefaction" (Lamb, §60). To obtain the *average energy density E* in the medium we must integrate $2T \cdot dx$ or $2V \cdot dx$ over an integral number of wave lengths and divide the result by the volume considered, thus

$$E \cdot \lambda = \int_x^{x+\lambda} 2T dx = \kappa' \int_x^{x+\lambda} s_0^2 \cos^2 \omega \left(t - \frac{x}{c}\right) dx$$

$$= \frac{\kappa' s_0^2}{2} \int_x^{x+\lambda} \left[1 + \cos 2\omega \left(t - \frac{x}{c}\right)\right] dx,$$

[1] The symbolic solution of the equation

$$\ddot{\xi} = c^2 \frac{\partial^2 \xi}{\partial x^2} \equiv c^2 \xi'' \tag{108}$$

leads quickly to the result $\dot{\xi} = \pm cs$ for the wave of velocity $\pm c$. We have, treating the dot and prime as operators,

$$\ddot{\xi} - c^2 \xi'' = (\dot{\xi} + c\xi')(\dot{\xi} - c\xi') = 0; \tag{108a}$$

whence

$$(\dot{\xi} + c\xi') = 0, \quad \text{or} \quad \dot{\xi} = -c\xi' = cs,$$

and

$$(\dot{\xi} - c\xi') = 0, \quad \text{or} \quad \dot{\xi} = c\xi' = -cs.$$

or noting that the integral of the periodic term vanishes (since $\frac{\omega\lambda}{c} = 2\pi$),

$$E\lambda = \tfrac{1}{2}\kappa' s_0{}^2\lambda, \quad E = \tfrac{1}{2}\kappa' s_0{}^2 \qquad (115)$$

in which s_0 is the maximum condensation.

The *Intensity* is defined as the rate of flow of energy past any fixed plane normal to the column of air. It is clearly equal to the product of the *energy density* and the *velocity of flow*, *c*. That is,

$$\left|\frac{dW}{dt}\right| = \frac{1}{2}(\kappa' s^2{}_{\text{max.}})c = \frac{1}{2}\frac{c}{\kappa'}(\delta p_{\text{max.}})^2 = \frac{1}{2}\frac{(\delta p_{\text{max.}})^2}{\rho c}, \qquad (116)$$

since
$$\delta p = \kappa' s. \qquad (105a)$$

This expression for *Intensity* in terms of *Excess Pressure* δp is a most useful one. As will appear later, it is valid for divergent as well as plane waves.

The *rate of working of the piston* in producing the outgoing sound wave is $\dot{\xi} \cdot \delta p$. Now since $\delta p = \kappa' s$ and $s = \frac{\dot{\xi}}{c}$ (112*a*), $\delta p = \frac{\kappa' \dot{\xi}}{c}$ and we have

$$\left|\frac{dW}{dt}\right|_{\text{piston}} = \dot{\xi} \cdot \delta p = \frac{\kappa'}{c}\dot{\xi}^2 = \sqrt{\kappa' \rho}\,\dot{\xi}^2 = R\dot{\xi}^2; \qquad (117)$$

also the simpler relation,

$$\delta p = \frac{\kappa'}{c}\dot{\xi} = \sqrt{\kappa' \rho}\,\dot{\xi} = R\dot{\xi}. \qquad (117a)$$

These important results define the quantity $R = \sqrt{\kappa' \rho}$ which is the *radiation resistance*[1] of the medium, *per unit area*. (*Cf*. § 8, Ch. I). We note that in the form

$$\sqrt{\kappa' \rho} \equiv R = \rho c, \qquad (117b)$$

[1] There is no consensus in naming this quantity. The engineer may prefer the term *characteristic impedance* of the medium; Drysdale gives *acoustic resistance*—a term due to Brillié (Etude des Ondes Acoustiques, etc., *le Génie Civil*, 75, 1919: Aug. 23, p. 171 Aug. 30, p. 194, and Sept. 6, p. 218). The term *radiation resistance* was suggested by, H. W. Nichols in *Phys. Rev.*, 10, 1917, p. 193.

it appears in many formulæ involving radiation; as for example in the radiation damping of a resonator (§ 24 and eq. 175). The tables for $\sqrt{\kappa' \rho}$ in Drysdale have already been mentioned.

One other property of plane waves, namely, *normal reflection* at the boundary between two media, should be dealt with. Let the plane $x = 0$ be the boundary between media (1) and (2). Choosing a solution of (108) in the form of (109) we have for waves in medium (1) to the left of the origin

$$\dot{\xi}_1 = \dot{\xi}_{01} e^{i\omega\left(t - \frac{x}{c_1}\right)} + \dot{\xi}'_{01} e^{i\omega\left(t + \frac{x}{c_1}\right)} \tag{118}$$

the two terms on the right representing respectively the particle velocities of oncoming and reflected waves. Similarly for the transmitted wave, in medium (2),

$$\dot{\xi}_2 = \dot{\xi}_{02} e^{i\omega\left(t - \frac{x}{c_2}\right)}. \tag{118a}$$

In order to determine the ratios $\frac{\dot{\xi}'_{01}}{\dot{\xi}_{01}}$ and $\frac{\dot{\xi}'_{02}}{\dot{\xi}_{01}}$ we require two equations of condition. These are available from the boundary conditions which are that both *velocity* and *excess pressure* must be continuous. (Some of the other quantities θ, (108) which satisfy the differential equation must also be continuous; but it is sufficient to deal with only two of them, the two chosen here leading most directly to the end in view).

We note that the velocities *are vectors*: the excess pressures (of the nature of hydrostatic pressures) strictly speaking are not. The phase difference (if any) between $\dot{\xi}_{01}$ and $\dot{\xi}'_{01}$ is as yet undetermined. To obtain the pressures we use the relation $\delta p = \pm c\rho\dot{\xi}$, paying particular attention to the sign before c, which depends on the direction of propagation; this is necessary because there can be no accumulation of uncompensated pressure at the boundary.

We have therefore, at the boundary

$$\dot{\xi}_{01} + \dot{\xi}'_{01} = \dot{\xi}_{02}, \tag{119}$$

i.e.,

or

$$\left.\begin{aligned} \delta p_{01} + \delta p'_{01} &= \delta p_{02}, \\ c_1\rho_1\dot{\xi}_{01} - c_1\rho_1\dot{\xi}'_{01} &= c_2\rho_2\dot{\xi}_{02}, \\ R_1(\dot{\xi}_{01} - \dot{\xi}'_{01}) &= R_2\dot{\xi}_{02}. \end{aligned}\right\} \tag{119a}$$

Treating (119) and (119*a*) as simultaneous equations, we obtain on solution [1]

$$\frac{\dot{\xi}_{02}}{\dot{\xi}_{01}} = \frac{2R_1}{R_1 + R_2} \quad \text{and} \quad \frac{\dot{\xi}'_{01}}{\dot{\xi}_{01}} = -\frac{R_2 - R_1}{R_2 + R_1}. \tag{120}$$

Now if $R_2 > R_1$, i. e., if the second medium is more resistant than the first, it is evident that in the reflected wave the particle velocity has undergone a phase change of π with respect to the particle velocity of the incident wave; but there is no phase difference between $\dot{\xi}_{01}$ and $\dot{\xi}_{02}$, the latter referring to the transmitted wave. It is also evident that while the particle velocity undergoes a phase change of π on reflection, the phase of excess pressure is unchanged. In the *transmitted wave* the phases of both the excess pressure and the particle velocity *are unchanged in any case.*

Again, if $R_2 < R_1$ the phase of the particle velocity is unchanged on reflection, while the excess pressure suffers a phase change of π. And finally, if $R_2 = R_1$ there *is no reflected* wave; transmission to the second medium is unimpaired.

The first ratio in (120) may be called a *transmission coefficient*, t_{12}; the second ratio is the (amplitude) *reflection coefficient*, r. Interchanging subscripts we have $t_{21} = \dfrac{2R_2}{(R_1 + R_2)}$, and the useful relation

in which

$$\left.\begin{aligned} t_{12}t_{21} &= 1 - r^2, \\ t_{12} &= \frac{2R_1}{R_1 + R_2}; \quad r = -\frac{R_2 - R_1}{R_2 + R_1}. \end{aligned}\right\} \tag{120a}$$

[1] Equations (120) are identical with the Fresnel Equations for normal optical reflection, if the radiation resistance R is replaced by the refractive index.

This is also obtainable by application of the energy principle to the incident, reflected and transmitted waves.

The classical example of acoustic reflection is in air, from a water surface; in this case [$R_2 = 1.4 \times 10^5$, $R_1 = 40$; $c_2 = 1.4 \times 10^5$, $c_1 = 3.3 \times 10^4$ (Drysdale's table)], so that

$$(\dot{\xi}'_{01})^2 = .999(\dot{\xi}_{01})^2.$$

From water to steel, the transmitted wave contains about 13 per cent of the original energy in the incident wave.

One other consequence is of interest. If the boundary to medium (1) is a (nearly) rigid wall, in which there is a *small* orifice containing a diaphragm (or other sound-detecting apparatus), we have, from (119*a*) and (120)

$$\delta p_{02} = \delta p_{01} + \delta p'_{01} \doteq 2\delta p_{01}, \text{ approximately.} \qquad (121)$$

from which we can calculate the motion of the diaphragm if we know its impedance.

32. *Sound Transmission in Tubes*

With the simple theory of the preceding sections we can solve (though not very elegantly) some of the problems of sound wave transmission in tubes. We shall have to make some restrictions to avoid mathematical difficulty; we shall assume, for example, that the walls of the tube are rigid,[1] and while we shall make some allowance for friction to add interest to the problem, we shall not specify the particular form of resistance

[1] This restriction is important, for if the contained medium has inertia and stiffness comparable with those of the wall material, the wall will yield appreciably to the excess pressure within. There are two consequences of this; first, the effective stiffness of the contained medium is diminished, hence a lowering of the wave velocity therein; and second, owing to dissipation in the wall itself, and lateral radiation from the wall, the wave in the contained medium suffers increased attenuation. This problem has been treated by Lamb (§62; also *Mem. Manchester Phil. and Lit. Soc.*, 42, No. 9, 1898, p. 1), who found (for example) that if water were inclosed in a glass tube whose thickness was one-tenth the radius, the phase velocity in the water would be diminished 24 per cent as compared with the normal value, on the basis of a rigid containing wall. An interesting experiment which the reader may try for himself will show the damping effect of thin-walled rubber tubing on sound waves transmitted through air in the tube.

coefficient to be used. For this particular phase of the problem the reader may consult Appendix A.

Choosing the tube of unit cross section, we derive the equation of wave motion anew, assuming that $R_1\dot{\xi}$ is equal to that component of the negative pressure gradient in phase with the velocity. R_1 is analogous to the ohmic resistance per unit length in an electrical transmission line: the whole problem is in fact parallel to the electrical problem, if there is no leakage between the pair of wires.

At any point x in the tube, let the excess pressure be $p = \kappa' s = -\frac{\kappa' \partial \xi}{\partial x}$; the net excess pressure acting on a lamina of thickness dx is therefore $dx \cdot \frac{\partial p}{\partial x} = - \kappa' dx \cdot \frac{\partial^2 \xi}{\partial x^2}$. This is partly compensated by the rate of change of momentum of the lamina, which is $\rho dx \cdot \ddot{\xi}$. The component of net excess pressure which is expended in doing work against friction is $dp_f = R_1 \dot{\xi} dx$ since $\frac{\partial p_f}{\partial x}$ is the effective pressure gradient.

Adding all these forces and equating the sum to zero, we have

$$\rho \ddot{\xi} dx + R_1 \dot{\xi} dx - \kappa' \frac{\partial^2 \xi}{\partial x^2} dx = 0, \tag{122}$$

or, as we prefer to deal entirely with velocity,

$$\rho \frac{\partial^2 \dot{\xi}}{\partial t^2} + R_1 \frac{\partial \dot{\xi}}{\partial t} = \kappa' \frac{\partial^2 \dot{\xi}}{\partial x^2}. \tag{122a}$$

This equation is identical, except for the friction term, with (108). (The analogous expression in the electrical case is

$$L \frac{\partial^2 i}{\partial t^2} + R \frac{\partial i}{\partial t} = \frac{1}{C} \frac{\partial^2 i}{\partial x^2}, \tag{122b}$$

the constants L, R and C being per unit length of line as in the acoustic problem.) (Fleming, "Propagation of Electric Waves," Third Edition (1919), p. 125.)

Assuming a solution $\xi = Ae^{i\omega\left(t - \frac{x}{c_1}\right)}$ we find on substitution

$$- \omega^2 \rho + iR_1 \omega = - \frac{\kappa' \omega^2}{c_1^2},$$

or

$$\frac{1}{c_1^2} = \frac{\rho}{\kappa'} - \frac{iR_1}{\omega\kappa'} = \frac{1}{c^2}\left[1 - \frac{iR_1}{\omega\rho}\right]; \tag{123}$$

and if $\frac{R_1}{\omega\rho}$ is a small quantity

$$\frac{1}{c_1} = \frac{1}{c}\left[1 - \frac{iR_1}{2\omega\rho}\right]. \tag{123a}$$

A complex value for the wave velocity appears as a natural result of dissipation in the transmission system, just as a complex value for natural frequency appears when damping is present in a vibrating system.

Placing the expression for $\frac{1}{c_1}$ in the assumed solution, we have, for a wave traveling in the positive direction

$$\left.\begin{aligned} \xi &= Ae^{i\omega\left(t - \frac{x}{c} + \frac{iR_1 x}{2\omega\rho c}\right)} = Ae^{-\alpha x} e^{i(\omega t - \beta x)}; \\ \beta &= \frac{\omega}{c}, \qquad \alpha = \frac{R_1}{2\rho c}. \end{aligned}\right\} \tag{124}$$

In equations of this sort, α is known as the *attenuation factor*; β, the *phase factor*, and if we like, $\alpha + i\beta$ is the *propagation constant*, as remarked previously (§26) in dealing with the iterated electrical structure. (We shall often use k for the phase factor, when $\alpha = 0$, i.e., when there is no attenuation due to friction). In (124) the *phase velocity* is $\frac{\omega}{\beta}$; if α is small (as here), the phase velocity is virtually the same as the unmodified wave velocity in the case of no dissipation. If α is large, however (as in the problems of §51, and Appendix A), the phase velocity will be appreciably less than $c \equiv \sqrt{\frac{\kappa'}{\rho}}$ for the free medium.

Now consider the phenomena in a finite tube of length l, closed at either end by a vibrating piston. For definiteness, let the piston at one end (taken as the origin) supply power of constant frequency to the system; at the other end the piston is driven by the excess pressure of the sound waves transmitted to and reflected by its surface. The boundary conditions are summarized thus:

	$x = 0$	$x = l$
Piston Impedance	Z_0	Z_l
Applied Force,	$\Psi = \Psi_0 e^{i\omega t}$	Zero
Velocity,	$\dot{\xi} = \dot{\xi}_0 e^{i\omega t}$	$\dot{\xi}_l e^{i\omega t}$
Pressure,	$p_0 e^{i\omega t}$	$p_l e^{i\omega t}$
Boundary Condition	$Z_0 \dot{\xi}_0 = \Psi_0 - p_0.$	$Z_l \dot{\xi}_l = p_l$

Owing to the recurrent reflection in the tube, there will travel in the positive direction from the origin a composite wave train, each component of which differs from the others only in having a different amplitude at the start, and a different phase constant depending on the number of times it has traversed the double length of the tube. These components form a series, of the type

$$\dot{\xi}_+ = \sum_{m=1}^{\infty} \cdot A_m e^{-(2ml + x)\alpha} \cdot e^{-i\left(\frac{2m\omega l}{c} + \beta x\right)} \cdot e^{i\omega t}$$

and it is clear that, since the amplitudes and phases of all the components follow the same law as they progress down the tube, they can all be summed up into one component of the same type, thus

$$\dot{\xi}_+ = A' e^{-(\alpha + i\beta)x} e^{i\omega t}$$

the constant A' being a carry-all for the series of coefficients each of which differs from its predecessor by constant attenuation and phase factors. Dealing similarly with the reflected wave system traveling in the negative direction from the point $x = l$, we write

$$\dot{\xi}_- = - B' e^{+(\alpha + i\beta)x} e^{i\omega t}$$

Consequently, for the velocity at any point in the tube

$$\dot{\xi}(x) = \dot{\xi}_+ + \dot{\xi}_- = (A'e^{-(\alpha + i\beta)x} - B'e^{(\alpha + i\beta)x})\, e^{i\omega t}. \quad (125)$$

The excess pressure p is $\pm\, c\rho\dot{\xi}$, the plus sign being used for the wave traveling in the positive direction, the minus sign for the reflected wave, as in the preceding section. Thus

$$p = \rho c(A'e^{-(\alpha + i\beta)x} + B'e^{(\alpha + i\beta)x})e^{i\omega t} \quad (126)$$

and considering maximum values only, we have

$$\left.\begin{aligned} &\text{at} \quad x = 0, \quad \dot{\xi}_0 = A' - B', \quad p_0 = \rho c(A' + B'); \\ &\text{at} \quad x = l, \quad \dot{\xi}_l = A'e^{-(\alpha + i\beta)l} - B'e^{(\alpha + i\beta)l}, \\ &\qquad\qquad\quad p_l = \rho c(A'e^{-(\alpha + i\beta)l} + B'e^{(\alpha + i\beta)l}). \end{aligned}\right\} \quad (127)$$

Now in view of the boundary conditions as to pressure at the piston faces, letting $L = e^{-(\alpha + i\beta)l}$ we have

$$\left.\begin{aligned} &Z_0(A' - B') + \rho c(A' + B') = \Psi_0, \\ &Z_l\left(A'L - \frac{B'}{L}\right) - \rho c\left(A'L + \frac{B'}{L}\right) = 0. \end{aligned}\right\} \quad (128)$$

The problem is thus reduced to that of a coupled system of two degrees of freedom; solving equations (128) for the constants A' and B', and putting these values in (125) and (126) we have

$$\left.\begin{aligned} &\dot{\xi}(x) = [(Z_l + \rho c)e^{-(\alpha+i\beta)x} - L^2(Z_l - \rho c)e^{(\alpha+i\beta)x}]\frac{\Psi_0}{D}e^{i\omega t}, \\ &\text{and} \\ &p(x) = \rho c[(Z_l + \rho c)e^{-(\alpha+i\beta)x} + L^2(Z_l - \rho c)e^{(\alpha+i\beta)x}]\frac{\Psi_0}{D}e^{i\omega t}, \\ &\text{in which} \\ &\quad L = e^{-(\alpha+i\beta)l}, \\ &\text{and} \\ &\quad D = (Z_0 + \rho c)(Z_l + \rho c) - L^2(Z_0 - \rho c)(Z_l - \rho c) \\ &\qquad = (Z_0Z_l + \rho^2c^2)(1 - L^2) + \rho c(Z_0 + Z_l)(1 + L^2). \end{aligned}\right\} \quad (129)$$

From this point a number of applications are possible. If we study the phenomena when the driving piston and the driven are electrically connected by means of an amplifier so that the motion of the piston at $x = l$ is enhanced and communicated to the driving piston, we are dealing with the "howling" telephone. This problem is interesting, but rather complicated, as it involves detailed study of the roots of $D = 0$ in terms of Z_0, Z_l and L in order to find the natural frequencies of the system. A suggestion as to theory is given by H. W. Nichols[1] (*Phys. Rev.* X, 1917, p. 191); if the reader wishes experimental details, these are available in Chap. XXIII of Kennelly's "Vibration Instruments." A complete analysis of the Howling Telephone Problem is given by H. Fletcher, in a paper in *Bell System Tech. Jour.*, V, Jan., 1926.

If we consider the steady state theory of wave transmission down the tube, we are dealing with the basis of the *Constantinesco wave-system of power transmission*, some of the properties of which are interesting and relatively easy to visualize. Those we shall note are of fundamental importance in any case.

Letting $x = 0$ in (129) we have for the *driving-point impedance*[2]

$$Z_{00} = \frac{\Psi_0}{|\dot{\xi}_0|_{\max.}} = \frac{(Z_0 + \rho c)(Z_l + \rho c) - L^2(Z_0 - \rho c)(Z_l - \rho c)}{(Z_l + \rho c) - L^2(Z_l - \rho c)}, \tag{130}$$

[1] The late H. W. Nichols (1886–1925) though known principally for his contributions to Electrodynamics, was an inspiring colleague and helpful critic in Acoustics as well—because of his keen interest in and thorough familiarity with the classical theory, which he often applied to practical problems. A biographical notice appeared in *Nature*, Dec. 19, 1925, p. 909.

[2] The form of (130), if dissipation is neglected, becomes

$$Z_{00} = \rho c \frac{Z_l \cos \beta l + i\rho c \sin \beta l}{\rho c \cos \beta l + iZ_l \sin \beta l} \tag{288}$$

as quoted in § 56. At this point the reader should familiarize himself (if he has not already done so) with Fleming's treatment of the parallel wire problem, in "The Propagation of Electric Currents," Chap. III. For example, (288) is identical with Fleming's equation (61), p. 99, if $\alpha = 0$. The reader will also be interested in comparing (288) with the equation obtained for the driving-point impedance of the exponential horn, by taking the ratio of (229) to (228), § 46; see also problem 35, following Chap. IV. In problems involving horns and tubes we have excellent examples of the value of impedance methods in studying acoustic systems.

Determining A and B, and substituting,

$$\xi(x) = \frac{\xi_0(e^{-i\beta x} - e^{i\beta x}e^{-2i\beta l})}{(1 - e^{-2i\beta l})} e^{i\omega t},$$

or
$$\xi(x) = \frac{\xi_0}{\sin \beta l} \sin \beta(l - x) \cdot e^{i\omega t}. \tag{135}$$

Thus "the amplitude becomes abnormally great, if $\sin \frac{\omega l}{c} = 0$, or $l = \frac{k\lambda}{2}$, k being integral" (Lamb, §62). The condition for tube resonance is exactly the same as in the Constantinesco scheme; but it must be recognized that the phenomena in the Kundt's tube are somewhat different. In Kundt's experiment there are standing waves whose kinematics are identical with those of the stretched string of § 28, p. 11; in the power transmission scheme, not all the energy is reflected at the distant end, owing to the yielding of the piston there. In this latter case there is in the tube a composite of standing wave conditions and wave transmission conditions if Z_0 and Z_l are imperfectly related; the *standing wave pattern tending to disappear when* Z_0 *and* Z_l have no reactance components, or reactance components which offset one another so that $Z_0 + Z_l$ is a pure resistance.

33. *Approximate Theory of Resonance in Tubes and Pipes*

We may now deal with some of the properties of organ pipes, which, in the simple theory are tubes with ends either wide open or rigidly closed. For closed ends, $Z_0 = Z_l = \infty$. In this case, again, equation (130) is not adapted to the discussion; it merely states, if we neglect ρc as relatively small, the undoubted (but useless) fact that $Z_{00} = Z_0$, a very large quantity. Since the tube with fixed ends cannot be driven from without, only the transient solution is of interest, and this is left as a problem for the reader, on the conclusion of this chapter.

If one or both ends of the tube are open we can obtain *transient solutions* on the basis of the *steady state theory* of equation (129) by allowing the piston impedances to vanish at the open

we have

$$Z_{00} = (Z_0 + Z_l), \qquad \text{if } \cos \beta l = \pm 1. \tag{133a}$$

We conclude that the impedance of the piston at $x = l$ is transferred (as if bodily) to the driving point where it is added to that of the directly driven piston. This is true with k either even or odd, i.e., with pistons exactly in or out of phase. The elastic medium furnishes in this ideal adjustment a massless, frictionless coupling of infinite stiffness between the two pistons: the ideal condition for wave power transmission.

The efficiency realized when the adjustment of frequency to tube resonance has been made will depend on minimizing the attenuation factor $\alpha = \frac{R_1}{2\rho c}\cdot$ For a given frequency, and medium of given viscosity, this depends on using the shortest possible tube, i.e., operating with tube one-half wave length long. In the Constantinesco system, however, long tubes are frequently necessary, with lateral connections at points along the tube from which the alternating pressure in the tube is used to drive a number of separate piston motors. It is obvious that in this case the higher harmonics of the tube must be called into play and that the lateral orifices must be disposed at points an *integral number of half wave lengths apart.* Those interested in the Constantinesco system will find further references in Drysdale, together with a practical discussion of the scheme.

In the familiar Kundt's tube experiment, Z_l is infinite and Z_0 is unknown, the boundary condition at the origin being simply that a *prescribed* motion $\xi_0 e^{i\omega t}$ takes place, this being due to the vibration at the end of a metal rod, itself in resonance. Equation (130) is therefore not adapted to the discussion. As in (125), we take (neglecting dissipation in the air column)

$$\xi(x) = (Ae^{-i\beta x} - Be^{i\beta x})e^{i\omega t}, \tag{134}$$

and we know that

$$\xi(0) = \xi_0 e^{i\omega t}; \qquad \xi(l) = 0.$$

Determining A and B, and substituting,

$$\xi(x) = \frac{\xi_0(e^{-i\beta x} - e^{i\beta x}e^{-2i\beta l})}{(1 - e^{-2i\beta l})}e^{i\omega t},$$

or
$$\xi(x) = \frac{\xi_0}{\sin \beta l}\sin \beta(l - x)\cdot e^{i\omega t}. \tag{135}$$

Thus "the amplitude becomes abnormally great, if $\sin \frac{\omega l}{c} = 0$, or $l = \frac{k\lambda}{2}$, k being integral" (Lamb, §62). The condition for tube resonance is exactly the same as in the Constantinesco scheme; but it must be recognized that the phenomena in the Kundt's tube are somewhat different. In Kundt's experiment there are standing waves whose kinematics are identical with those of the stretched string of § 28, p. 11; in the power transmission scheme, not all the energy is reflected at the distant end, owing to the yielding of the piston there. In this latter case there is in the tube a composite of standing wave conditions and wave transmission conditions if Z_0 and Z_l are imperfectly related; the *standing wave pattern tending to disappear when* Z_0 *and* Z_l have no reactance components, or reactance components which offset one another so that $Z_0 + Z_l$ is a pure resistance.

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If one or both ends of the tube are open we can obtain *transient solutions* on the basis of the *steady state theory* of equation (129) by allowing the piston impedances to vanish at the open

ends. This is given as an interesting variation from the standard textbook method. But as in the classical treatment we must point out just what is involved in neglecting terminal impedance at an open end.

The radiation resistance of a source of area S (cf. §24) has been given as

$$b_1 = \frac{\rho\omega^2 S^2}{4\pi c} \text{ in terms of frequency.}$$

(This equation will appear as (175) later in this chapter.)

If we take $S = 1$, b_1 becomes for all moderate values of frequency appreciably smaller than ρc, the radiation resistance for plane waves; thus no great harm is done in neglecting b_1. But this is not all; there is an inertia component of the impedance at an open end the effect of which is virtually to increase the length of the tube by a small increment. This is a complicated matter the discussion of which we shall defer to Chap. IV; in the present situation we shall assume that ρc is much greater than the impedance of an open end, and obtain what information we can on this basis.

The simplest case to discuss is that for which $Z_0 = 0$ (i.e., $Z_0 < \rho c$) and Z_l is very great. Making this change in equation (129), we have for the velocity, neglecting dissipation,

$$\dot{\xi}(x) = \frac{Z_l(e^{-i\beta x} - L^2 e^{i\beta x})\Psi_0 e^{i\omega t}}{\rho c Z_l(1 + L^2)} \tag{136}$$

$$= \frac{[e^{-i\beta x}e^{i\beta l} - e^{-i\beta l}e^{i\beta x}]\Psi_0 e^{i\omega t}}{\rho c(e^{i\beta l} + e^{-i\beta l})}, \tag{136a}$$

whence

$$\dot{\xi}(0) = \frac{i}{\rho c}\frac{\sin \beta l}{\cos \beta l}\Psi_0 e^{i\omega t}. \tag{136b}$$

Thus the driving-point impedance is a pure reactance

$$(-i\rho c \cot \beta l)$$

and the natural frequencies are found by placing this quantity equal to zero. Applying this condition we have

$$\left.\begin{aligned} &\cos \beta l = 0, \\ \text{whence} \quad & \\ &\beta l = k\frac{\pi}{2} \qquad (k = 1, 3, 5, \ldots) \\ \text{that is,} \quad & \\ &\omega_k = \frac{kc\pi}{2l}, \\ \text{or since} \quad & \\ &c = \lambda f, \qquad l = \frac{k\lambda}{4}. \end{aligned}\right\} \tag{137}$$

The velocity is of course very great for these frequencies, for which the tube is an odd number of quarter wave lengths long.

Since the denominator is not a function of x, we can take the numerator of (136a) as indicating the distribution of velocity in the tube, under resonance conditions. We have then $e^{i\beta l} = i$ and $e^{-i\beta l} = -i$ and we may write

$$\dot{\xi}(x) = \dot{\xi}_0 \cos \beta x \cdot e^{i\omega t} \equiv \dot{\xi}_0 \cos \frac{k\pi x}{2l} \cdot e^{i\omega_k t} \tag{138}$$

in which $\dot{\xi}_0$ is the maximum value of $\dot{\xi}$ for $x = 0$. To obtain the pressure distribution along the tube we cannot use the relation $\delta p = \pm c\rho\dot{\xi}$, because the waves in both positive and negative directions have been combined to obtain $\dot{\xi}(x)$ as in (138). But we always have

$$p = \kappa' s = -\kappa'\frac{\partial \xi}{\partial x} = -c^2\rho\frac{\partial \xi}{\partial x} \qquad (105a) \text{ and } (113)$$

whether the wave system is standing or moving. Thus, since $\xi(x) = \dfrac{\dot{\xi}}{i\omega}$,

$$p = -c^2\rho\frac{\partial}{\partial x}\left(\frac{\dot{\xi}_0}{i\omega}\cos \beta x\right)e^{i\omega t} = -i\rho c\dot{\xi}_0 \sin \beta x \cdot e^{i\omega t} \tag{139}$$

and it appears that the pressure is zero at the point $x = 0$, and

a maximum at $x = l$; the *pressure* distribution in the *standing wave system* being *complementary* to the *velocity* distribution along the tube. Moreover, the maxima of pressure are out of phase with the maxima of velocity by one-quarter of a period; but pressure and amplitude are in phase as far as *time* is concerned. These relations are generally true in connection with standing sound wave phenomena; the reader may note particularly the contrast between this state of affairs and the progressive wave phenomena emphasized in connection with the discussion of energy, following equation (112*a*).

For the tube open at both ends ($Z_0 = Z_l = 0$) the velocity is, according to equation (129), (if dissipation is neglected)

$$\dot{\xi}(x) = \frac{\rho c(e^{-i\beta x} + L^2 e^{i\beta x}) \cdot \Psi_0 e^{i\omega t}}{\rho^2 c^2(1 - L^2)} \tag{140}$$

$$= \frac{(e^{-i\beta x}e^{i\beta l} + e^{i\beta x}e^{-i\beta l})}{\rho c(e^{i\beta l} - e^{-i\beta l})}\Psi_0 e^{i\omega}, \tag{140a}$$

whence

$$\dot{\xi}(0) = \frac{\cos \beta l}{i\rho c \sin \beta l}\Psi_0 e^{i\omega t} \quad \text{and} \quad \dot{\xi}(l) = \frac{\Psi_0 e^{i\omega t}}{i\rho c \sin \beta l}. \tag{140b}$$

The driving-point impedance is now $i\rho c \tan \beta l$, and the natural frequencies are given by

$$\beta l = k\pi, \quad (k = 1, 2, 3, \ldots) \quad \text{or} \quad l = \frac{k\lambda}{2}; \tag{141}$$

a well-known result for the "open" pipe, usually obtained in a more direct way. The velocity and pressure distributions in the pipe may be obtained by the method applied to the tube closed at one end, in which event equations similar to (138) and (139) will be obtained; this is left to the reader. In view of the discussion that has been given it should not be necessary to deal further with the mechanics of resonance in tubes.[1]

[1] The behavior of a cylindrical pipe in which lateral holes have been cut is treated theoretically by W. Steinhausen, *Ann. d. Phys.*, 48, 1915, p. 693, and experimentally by E. Ratz, *Ann. d. Phys.*, 77, 1925, p. 195. Incidentally Ratz determines the end correc-

To give an exact account of the phenomena in tubes, dissipation would have to be included; but the pressure and velocity relations in standing wave systems which we have obtained will not be greatly affected thereby.

34. *General Discussion of the Physical Factors Affecting Transmission*

Before going further with the general theory we must consolidate the position already gained (as was done in § 8) by reviewing certain physical factors affecting transmission. These phenomena can be treated only in barest outline; there is a fairly complete mathematical theory of them, but it is too lengthy and involved to include in an elementary course.

Consider absorption first: this may take two forms. If due to resonance it will be selective, and the theory of the reaction of resonant structures on sound-waves striking them is not what we have in mind at present. There are many problems of this sort, but they are more properly classed as radiation problems. What we have in mind is inherent absorption due to friction in the medium, and resulting in the extinction of a progressive wave, after a certain distance of the medium has been traversed. This is non-selective, except as "constants" of the medium vary somewhat with frequency, and was illustrated to a marked degree in the propagation of compressional waves in the thin damping film of the air-damped transmitter. [§ 11, eq. (37).]

We now wish to apply the theory of dissipation to sound transmission in conduits of capillary dimensions. A solution of this problem is given in Appendix A; for very narrow tubes it appears that Poiseuille's law is applicable and that the inertia of the fluid in the tube is negligible as compared with the viscosity,

tion for an (unflanged) pipe to be $\alpha = .78r$; this is between the figures given by Rayleigh for a flanged and an unflanged end. (See § 43, Chap. IV.)

In Rayleigh (II), § 318) a discussion is given of the absorption of energy from sound waves transmitted through a tube, when they pass a lateral opening leading to a resonator. The reader interested in phenomena of this kind may refer to three recent papers by G. W. Stewart on the Effect of Branches on Acoustic Transmission through a Conduit: see *Phys. Rev.*, 25, 1925, p. 688; *ibid.*, 27, 1926, p. 487 and p. 494.

as far as the effect on transmission is concerned. The equation of motion may be written

$$\dot{\xi} = -\frac{1}{R}\frac{\partial p}{\partial x} = \frac{\kappa'}{R}\frac{\partial^2 \xi}{\partial x^2}, \qquad R = \frac{8\mu}{r^2}, \qquad (K')$$

the solution of which shows that such waves as are transmitted (or rather diffused) into a capillary tube are very highly damped.

It is natural to seek in these phenomena an explanation of the absorption of sound by porous bodies. The pores or crevices in the absorbing material are filled with air and therefore present to an incident sound-wave an array of open ends of narrow conduits within which dissipation due to viscosity is very high. The problem of producing a good absorbing material is really twofold. We must have (to prevent reflection, § 31) a substance whose radiation resistance is as nearly as possible equal to that of the adjacent medium from which the sound is to be absorbed; and in addition we must provide in the material as much friction as possible to extinguish the transmitted wave. Considering felt (for example), if it is packed too loosely, while its radiation resistance will nearly match that of the air (due to its high percentage content of air), the air spaces in the material will not be sufficiently constricted to bring viscosity forcibly into play, and a great thickness of the absorbing material is required to produce extinction of the transmitted wave. If packed too tightly, there is ample friction available, but the felt now has too much inertia, its radiation resistance is too high as compared with air, and consequently the sound waves are reflected without penetrating the medium at all. The deduction as to what constitutes a good absorbing material should be clear. We shall consider the theory of absorbing materials, on the basis sketched, in greater detail in Chap. V (§ 51, § 52).

While it is not our purpose to go into the properties of specific materials, mention should be made of a neat and effective method used by E. C. Wente for testing the absorbing properties of various kinds of felt. A short tube (of sufficient diameter to minimize internal friction) was driven by a telephone re-

ceiver at one end; the other end was closed by a disc of the absorbing material backed by a rigid disc. The conditions of the experiment are exactly those of the wave-transmission scheme which we have considered at length (§ 32); the standing wave pattern (if the frequency is adjusted to tube resonance) tends to disappear as more and more of the energy sent out by the telephone is absorbed at the distant end of the tube. (The standing wave pattern in the tube is determined by means of a small exploring tube connected to a sound measuring device, such as a condenser transmitter). The experiment can be performed in various ways; for a *fixed* length of the tube *in resonance*, we can determine the relation between the pressure maxima and minima of the standing wave system, and the reflection coefficient of the absorbing surface, following the idea of H. O. Taylor, *Phys. Rev.*, II, 1913, p. 270. This procedure would be more or less along classical lines. (Another variation of the experiment is illustrated in problem 41, p. 226.) But in practice Wente prefers to measure the *driving-point impedance* of the tube, *when its length is varied*; this will be a maximum when the effective length of the tube is an integral number of half wave lengths (cf. p. 102) and a minimum when this adjustment of length is changed by a quarter wave length. From these measured impedances the absorption coefficient of the layer can be calculated.

We consider now the milder forms of dissipation, in an infinite medium, which tend to degrade sound waves. The degrading influences include everything which may change the ordered oscillatory motion ξ of the particles of the medium into the disordered or statistical motions associated with the heat content of the medium. These influences are particularly enhanced in large-scale phenomena, such as, for example, the propagation of a pressure impulse originating in a distant explosion; while the local effect on the wave of departures from ideal conditions in any one part of the medium may be moderate, the cumulative effect during the passage of the wave from beginning to end may be considerable. These effects, while not of fundamental importance to those engaged in the study of

small-scale acoustic phenomena (as applied, for example, in telephony) should at least be considered.

A typical degrading influence is viscosity, even if there are no constricting boundaries. No substance can be expanded or contracted, in one dimension, without dissipation; or to put the matter another way, sound waves can traverse no medium, however light or elastic, without absorption to some degree. In the case of a *gaseous medium* it can be shown (Lamb, § 64) that the general equation for the propagation of a plane wave is:

$$\rho \frac{\partial^2 \xi}{\partial t^2} = \kappa' \frac{\partial^2 \xi}{\partial x^2} + \frac{4}{3} \mu \frac{\partial^2 \dot{\xi}}{\partial x^2}, \tag{142}$$

in which μ is the viscosity coefficient. This equation is reminiscent of (122); the dimensions of the friction terms are exactly the same in the two cases and the solution is obtained in a similar way. Equations (142) and (122) are identical if we write

$$R_1 \dot{\xi} = -\frac{4}{3} \mu \frac{\partial^2 \dot{\xi}}{\partial x^2} = \frac{4}{3} \frac{\mu \omega^2 \dot{\xi}}{c^2}, \tag{143}$$

as we may, because $\frac{\partial^2 \dot{\xi}}{\partial x^2} = -\frac{\omega^2 \dot{\xi}}{c^2}$ for a progressive wave motion. Then taking as the solution of (142),

$$\xi = Ae^{-\alpha x} e^{i\omega\left(t - \frac{x}{c}\right)} \quad \text{(cf. 124)} \tag{144}$$

we have for the attenuation factor

$$\alpha = \frac{R_1}{2\rho c} = \frac{2\mu\omega^2}{3\rho c^3} = \frac{2}{3} \frac{\mu k^2}{\rho c}, \tag{144a}$$

from (143). The coefficient $\frac{\mu}{\rho} = \nu$ has a special name; it was called the "*kinematic*" *coefficient of viscosity* by Maxwell. For air it is about .132 cm.^{2}/sec. at normal temperature and pressure. The attenuation coefficient α is small, judged by the standards of small-scale phenomena in the laboratory; but it rises rapidly

with frequency. Thus, foghorn signals, when heard over great distances, arrive with the fundamental tones relatively accentuated at the expense of the higher frequency components. (This effect incidentally is heightened by the "eddy structure" of the atmosphere, another large-scale hazard that tends to degrade sound waves. See L. V. King, *Phil. Trans.*, 218A, p. 211, 1919, and G. I. Taylor, *ibid.*, 215 A, p. 1, 1915.)

Equation (142), which is due to Stokes, was derived for a laterally unlimited isotropic medium. In the case of a gas, $\kappa' = \gamma p$, that is, the effective bulk modulus for rapid vibrations. In the case of longitudinal waves in a solid bar, κ' must be replaced by Young's Modulus (Lamb, § 43), and in the dissipative term the factor μ must be multiplied by $(1 + \sigma)$, σ being Poisson's ratio of lateral contraction to longitudinal extension. A derivation of the equation in the form described is given by S. L. Quimby, *Phys. Rev.*, 25, 1925, p. 558.

An unpublished experiment of P. W. Bridgman and H. M. Trueblood (1918) is of interest in this connection. They wished to determine the phase velocity of compressional waves excited in a column of rubber, when one end of the column was connected normally to a vibrating telephone diaphragm. The attenuation due to viscosity in the rubber is very rapid, and there was no reflected wave from the distant end. The state of the progressive wave was investigated by means of a light microphone element attached at various points; and it appeared that, from the phase of the vibration of the exploring microphone, the wave length and hence the velocity of the sound in the rubber (column) could be determined: this was of the order of 80 metres/sec. for a certain sample. Theoretically it should have been possible, by observing the diminution in intensity with distance along the column, to determine the attenuation constant, α, and from this to compute the coefficient of viscosity of the rubber for the frequency of excitation of the system. Unfortunately the experimental data did not permit this to be done with any accuracy, but the data did show the variations in viscosity from sample to sample, and a general effect of decreasing viscosity as the frequency was raised. This is quali-

tatively in agreement with a point discussed by Quimby (loc. cit.) in dealing with the viscosity of glass and metals under the rapidly alternating strains due to longitudinal waves. There is every reason to expect that the effective viscosity of the material will vary with frequency, and that the viscosity under high-frequency excitation will be quite different from the statically determined constant. Quimby found, for example, viscosity coefficients of the order of 10^3 for glass, aluminum, and copper at frequencies of the order of 40,000, whereas the statically determined constants are in the neighborhood of 10^8. The method used by Quimby depends on the standing wave phenomena in bars driven into longitudinal vibration, and is in itself an interesting application of the theory of acoustic systems.[1]

Heat conduction (i.e., failure of the laminar elements in the medium to expand or contract adiabatically) is also a cause of degradation. This was investigated by Kirchoff (cf. Lamb, § 65) who found that the attenuating effect was comparable to that accounted for by viscosity.

When sound waves are of finite (i.e., large) amplitude, as distinguished from the usually considered condition of "small oscillations," degradation results to a marked degree. The wave velocity depends on the *amplitude*; the medium in effect does not obey Hooke's Law. In the classical theory (Lamb, § 63) it is shown that the "crests" are propagated with slightly higher velocity than the "troughs"; the former tend to overtake the latter in transit and distortion of wave form is inevitable. This case is of considerable importance to those working in the field of large-scale acoustic transmission, and some recent experimental work by M. D. Hart (*Proc. Roy. Soc.* 105A, p. 80, Jan. 1, 1924) may interest the reader in this connection. It is evident that if the "crests" tend to pile up and become steeper,

[1] The paper also gives an interesting discussion of the discrepancies between Stokes theory and the observed data, arising when viscosity coefficients or displacements are excessive. Again, the piezo-electric driving system is of special interest in connection with other piezo-electric applications referred to in Appendix B.

Viscosity under alternating stresses is also discussed by R. W. Boyle, (*Trans. Roy. Soc. Canada*, Ser. III, 16, 1922, p. 293) "Compressional Waves in Metals."

a point will ultimately be reached at which disordered motion takes the place of a certain part of the original acoustic energy.

To all the effects mentioned above there are added in any practical case, meteorological hazards, such as refraction due to varying temperature gradients in the atmosphere, or to varying air currents. There are analogous disturbing factors in submarine signalling, though not to so great a degree; and it may be noted that the inherent losses in water (which depend on the kinematic viscosity) are less than in air. It is indeed fortunate that in telephony as ordinarily practiced no great use is made of acoustic phenomena of the large-scale kind.[1]

35. *General Theory of Sound Waves in Three Dimensions*

In most of the problems of sound radiation *divergence* plays a dominant part; consequently we must extend the theory which we have used for plane waves to include motion in three dimensions. Instead of taking a thin lamina of the medium (§ 30) we consider a small element of volume $\delta x \cdot \delta y \cdot \delta z$. The net force in the x-direction due to the excess pressure on this element is

$$-\frac{\partial p}{\partial x}\delta x \cdot dydz,$$

and since the force due to inertia is $\rho\ddot{\xi} \cdot dxdydz$ we have (cf. 107)

$$\rho\ddot{\xi} + \frac{\partial p}{\partial x} = 0 \qquad (145)$$

for the equation of motion along the x-axis. If η and ζ are respectively the displacements in the y and z directions, we have the set of equations

$$\rho\ddot{\xi} = -\frac{\partial p}{\partial x}, \quad \rho\ddot{\eta} = -\frac{\partial p}{\partial y}, \quad \rho\ddot{\zeta} = -\frac{\partial p}{\partial z}. \qquad (145a)$$

[1] The reader specially interested in sound signalling in air, meteorological hazards, etc., is referred to the monograph on "Principles of Sound Signalling," by M. D. Hart and W. Whately Smith: London, 1925. (Reviewed in *Nature*, 116, Dec. 12, 1925.)

In these equations p represents the excess pressure: and therefore $p = \kappa' s$ (eq. 105a); also we may put $c^2 = \dfrac{\kappa'}{\rho}$, so that the set (145$a$) may be replaced by

$$\ddot{\xi} = -c^2\frac{\partial s}{\partial x}, \quad \ddot{\eta} = -c^2\frac{\partial s}{\partial y}, \quad \ddot{\zeta} = -c^2\frac{\partial s}{\partial z}. \qquad (145b)$$

It is not an economy to deal with three separate equations of motion in a case like this, any more than it would be in an ordinary oscillation problem to use $3m$ equations for a system of m degrees of freedom (cf. § 20). To unify the theory we seek a single function from which all three of the quantities ξ, η, ζ (or their time derivatives) can be derived. Such a function *exists* for all cases of fluid motion in which there is *no circulation* (e.g., no *eddies* or *vortices*) and this is plainly the state of things when the particles of the medium execute small oscillations in transmitting sound waves. The function we are seeking is the *Velocity Potential*, denoted by ϕ, and is related [1] to the velocities in any hydrodynamical problem as follows:

$$\dot{\xi} = -\frac{\partial \phi}{\partial x}, \qquad \dot{\eta} = -\frac{\partial \phi}{\partial y}, \qquad \dot{\zeta} = -\frac{\partial \phi}{\partial z}. \qquad (146)$$

For this fundamental idea we are again indebted to Lagrange.

(An alternative definition to (146) is

$$\phi = -\int(\dot{\xi}dx + \dot{\eta}dy + \dot{\zeta}dz), \qquad (146a)$$

[1] The reader will note that throughout we follow the convention of a negative sign in such equations as

$$\dot{\eta} = -\frac{1}{R}\frac{dp}{dr} \ (36); \qquad \rho\ddot{\xi} = -\frac{\partial p}{\partial x} \ (145a); \qquad \dot{\xi} = -\frac{\partial \phi}{\partial x}. \qquad (146)$$

This usage is in accord with that of Lamb, but contrary to that of Rayleigh. It seems to the writer only common sense to use the negative sign because the natural motions or fluxes which take place in any field of force are from points of high pressure, or high potential, to points of lower pressure or potential; in other words, the directional derivative or gradient must be negative with respect to the corresponding motion. If further justification is required for this usage, the remarks of Lamb in the Preface to his "Hydrodynamics" are to the point.

which is analogous to the line integral of the forces, used in computing the work done, to obtain the Newtonian potential function in mechanical problems; but (146*a*) is not as useful in the present instance as (146). The function ϕ is a scalar, just as the mechanical potential function V is a scalar; but (like V) its directional derivatives are vectors. The dimensions of ϕ are L^2T^{-1}.)

To obtain ϕ we first integrate (145*b*) obtaining

$$\dot{\xi} = -c^2\frac{\partial}{\partial x}\int_0^t s dt + \dot{\xi}_0, \quad \dot{\eta} = -c^2\frac{\partial}{\partial y}\int_0^t s dt + \dot{\eta}_0,$$

$$\dot{\zeta} = -c^2\frac{\partial}{\partial z}\int_0^t s dt + \dot{\zeta}_0, \tag{147}$$

the constants $\dot{\xi}_0$, $\dot{\eta}_0$, $\dot{\zeta}_0$ being the original component velocities, when $t = 0$. If we take for these constants $\dot{\xi}_0 = -\dfrac{\partial \phi_0}{\partial x}$, etc., we have, on comparison of (147) and (146)

$$\phi = c^2\int_0 s dt + \phi_0, \tag{148}$$

which is the velocity potential function sought. (148) yields at once the relation

$$\dot{\phi} = c^2 s \equiv \frac{p}{\rho}, \tag{149}$$

which is important in several ways as will appear.

One application of (149) is immediate. We have given the equation of continuity (106, 106*a*) in one dimension, and by an easy extension of the idea to three dimensions, we have

$$\frac{\partial s}{\partial t} + \frac{\partial \dot{\xi}}{\partial x} + \frac{\partial \dot{\eta}}{\partial y} + \frac{\partial \dot{\zeta}}{\partial z} = 0. \tag{150}$$

Writing for s, $\dfrac{\dot{\phi}}{c^2}$ and for $\dot{\xi}$, $-\dfrac{\partial \phi}{\partial x}$ and so on for the other two velocities, we have

$$\frac{\partial^2 \phi}{\partial t^2} - c^2\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right) = 0, \tag{151}$$

or, using the accepted abbreviation $\nabla^2\phi$ for the expression in parentheses,

$$\ddot{\phi} - c^2\nabla^2\phi = 0, \tag{151a}$$

which is the general equation for the propagation of sound waves in three dimensional space.

The general procedure in solving a three-dimensional problem is as follows: we first find a solution of (151*a*) which satisfies the boundary conditions; knowing ϕ we can determine the excess pressure p, the condensation s, or any other variable in which we are interested from the relations (149) and (146).

36. *Spherical Waves of Sound; the Point Source*

Some of the simpler properties of spherical waves of sound may now be investigated. In spherical coordinates we may write

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \tag{152}$$

Now if there is spherical symmetry about the origin (the case of a small source of sound at $x = 0$, $y = 0$, $z = 0$) ϕ is a function of only r and t and (151*a*) becomes

$$\ddot{\phi} - c^2\left(\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}\right) = 0 \tag{153}$$

or, if a new function $\Phi = r\cdot\phi$ is chosen, we have instead of (153)

$$\frac{\partial^2}{\partial t^2}(r\phi) = c^2\frac{\partial^2}{\partial r^2}(r\phi). \tag{153a}$$

This equation is kinematically equivalent to (108) if we let $\theta = (r\phi)$, consequently it must have a solution of the type

$$\phi = \frac{A'}{r}f(ct - r) + \frac{B'}{r}F(ct + r), \tag{154}$$

the first term of which represents a spherical wave diverging radially from the source, and the second a spherical wave in the reverse direction just as we noted for equations (108) and (109) for plane waves. A and B are arbitrary constants to which an interpretation is to be given. It is evident at once that all expressions for the energy in the wave will contain ϕ^2 or $\frac{A^2}{r^2}$ so that the *energy density* at any point distant r from the origin will vary as $\frac{1}{r^2}$; and the energy flow through any spherical surface surrounding the origin must be a constant, independent of r, consistently with the energy principle.

Fixing attention on only the divergent wave, the first relation we consider is that between the particle velocity $\dot{\xi}(r)$ and the excess pressure. Since $\dot{\xi}(r) = -\frac{\partial\phi}{\partial r}$, and $p(r) = \rho\dot{\phi}$ (149) we have

$$\left.\begin{aligned} \dot{\xi}(r) &= +\frac{A'}{r}f'(ct-r) + \frac{A'}{r^2}f(ct-r), \\ \text{and} \qquad p(r) &= \frac{\rho c A'}{r}f'(ct-r), \end{aligned}\right\} \tag{155}$$

$f'(ct-r)$ denoting the derivative of f with respect to $(ct-r)$. At appreciable distances from the origin the second term in the first equation becomes small as compared with the first term, so in any practical case

$$p = \rho c\dot{\xi} = R\dot{\xi} \qquad (r \text{ large}). \tag{156}$$

Also, since in any small element of volume $\delta r \cdot r\delta\theta \cdot r\sin\theta \cdot \delta\phi$ we must have $p = \kappa' s$ (p and s varying only with r)

$$cs = \dot{\xi} \quad \text{and} \quad p = \frac{\kappa'}{c}\dot{\xi}. \tag{156a}$$

These are all relations which we have found for plane waves; the only distinction between the two cases (if r is not too small) is that we must consider $(r\dot{\xi})$ and (rp) for the divergent wave in

place of $\dot{\xi}$ and p for the plane wave. This merely allows for the decrease in energy density due to divergence.

It can also be shown that the maxima of kinetic and potential energy at any point are equal, as in the case of plane waves; the same remarks as to energy and phase that were made following (112*a*) in § 31 apply here.

Conditions *at the source* are especially interesting. Suppose fluid is introduced periodically there so that

$$\phi = \frac{A'}{r} f(ct - r) \equiv \frac{A'}{r} \cos \omega\left(t - \frac{r}{c}\right); \tag{154a}$$

then

$$\dot{\xi}(r) = -\frac{A'\omega}{rc} \sin \omega\left(t - \frac{r}{c}\right) + \frac{A'}{r^2} \cos \omega\left(t - \frac{r}{c}\right), \tag{155a}$$

and the total *flux* (or rate of flow of fluid) through a spherical surface of radius r is

$$4\pi r^2 \dot{\xi}(r) = -\frac{4\pi A'\omega}{c} r \sin \omega\left(t - \frac{r}{c}\right) + 4\pi A' \cos \omega\left(t - \frac{r}{c}\right). \tag{157}$$

Now let r_0 be the radius of the source, which we can take as a very small sphere. Then if $\dot{\xi}(r_0) = \dot{\xi}_0 \cos \omega t$, and if we take

$$(4\pi r_0^2 \dot{\xi}_0) \cos \omega t = A \cos \omega t \tag{158}$$

we have in the limit, as $r \doteq r_0 \doteq 0$, from (157)

$$A \cos \omega t = 4\pi A' \cos \omega t. \tag{159}$$

The quantity $A = 4\pi r_0^2 \cdot \dot{\xi}_0$ is *the maximum rate of emission of fluid at the source* and is called the *strength of the source*. We have thus determined the constant A' of (154*a*) and we may write for the velocity potential at any point in the medium due to a small isolated source whose strength is A, and whose time periodic factor is $\cos \omega t$,

$$\phi = \frac{A}{4\pi r} \cos \omega\left(t - \frac{r}{c}\right) \equiv \frac{A}{4\pi r} \cos (\omega t - kr). \tag{160}$$

In making frequent use of this expression we shall be concerned not so much with the exact shape of the source, as with its size.

If it is irregular in shape, or flat (as, for example, one side of a piston), it is important that it be small as compared with the wave length, or (160) will not be applicable, due to the essential lack of spherical divergence from the source. But it may be noted that in many problems where we plainly do not have spherical divergence from the source as a whole, we can apply an expression similar to (160) to individual *elements* of the source and by an integration obtain the velocity potential at any point in the field, due to the source as a whole.

The rate of working of the system of sound waves from the simple source must be the same at any spherical surface surrounding the source. It is clearly obtainable as the product of the flux and the pressure at any such surface. We have from (160), for the velocity and pressure

$$\dot{\xi} = -\frac{\partial\phi}{\partial r} = -\frac{Ak}{4\pi r}\sin(\omega t - kr) + \frac{A}{4\pi r^2}\cos(\omega t - kr) \quad (161)$$

and

$$p = \rho\dot{\phi} = -\frac{A\omega\rho}{4\pi}\sin(\omega t - kr). \quad (161a)$$

The rate of working is therefore

$$4\pi r^2\cdot\dot{\xi}\cdot p = \frac{A^2 k\omega\rho}{4\pi}\left[\sin^2(\omega t - kr) - \frac{1}{kr}\sin(\omega t - kr)\cos(\omega t - kr)\right]$$

$$= \frac{\rho c\cdot A^2k^2}{8\pi}\left[1 - \cos 2(\omega t - kr) - \frac{1}{kr}\sin 2(\omega t - kr)\right]; \quad (162)$$

and we have for the *average rate at which work is done by the source*

$$4\pi r^2\cdot\frac{dW}{dt} = \rho c\cdot\frac{A^2k^2}{8\pi}, \quad (163)$$

in which the quantity

$$\left|\frac{dW}{dt}\right|_{\text{av.}} = \overline{\dot{\xi}\cdot p} = \rho c\cdot\frac{A^2k^2}{32\pi^2 r^2}, \quad (164)$$

is evidently the *sound intensity* at the surface of a sphere of radius r, due to the source of strength A at the origin. On the

principle of equation (116) the *energy density* in the medium must be

$$E = \frac{1}{c}\frac{dW}{dt} = \rho\frac{A^2k^2}{32\pi^2r^2}. \tag{165}$$

This is obviously equivalent to

$$E = \frac{1}{2}\kappa' s^2_{\text{max.}} = \frac{1}{2}\frac{p^2_{\text{max.}}}{\kappa'}, \tag{166}$$

as in the case of plane waves; and again we have for the intensity the useful formula

$$\left|\frac{dW}{dt}\right|_{\text{av.}} = \frac{1}{2}\frac{p^2_{\text{max.}}}{\rho c}, \tag{164}$$

identical with (116) for plane waves.

37. *The Pulsating Sphere as a Generator of Sound*

The most interesting and instructive problem to which the theory can be immediately applied is that of determining the reactions on a pulsating sphere of *finite* radius, used as a sound generator in the unlimited medium. Since we are not dealing with a point source, we cannot neglect the second term of the expression of the particle velocity in (155*a*), and this as will appear is an important element in the consideration of the problem. It is desirable to make a new start, using complex quantities for brevity in all operations.

The first boundary condition is that when $r = r_0$ (the radius of the sphere) the velocity shall be (the real part of)

$$\dot{\xi}(r_0) = \dot{\xi}_0 e^{i\omega t}. \tag{167}$$

We must therefore take, for the solution of the differential equation (153*a*), for the divergent wave only, a function of the form

$$\phi = \frac{A'}{r}e^{i[\omega t - k(r - r_0)]}. \tag{168}$$

A second boundary condition, namely, that the velocity shall vanish at infinity is obviously satisfied. From (168) we have, for the velocity at the surface of the pulsating sphere

$$\dot{\xi}(r_0) = -\left.\frac{\partial\phi}{\partial r}\right|_{r=r_0} = A'\frac{(1 + ikr_0)}{r_0^2}e^{i\omega t} \tag{169}$$

which must be identical with (167); thus comparing coefficients

$$A' = \frac{r_0^2}{1 + ikr}\dot{\xi}_0 = \frac{r_0^2(1 - ikr_0)}{1 + k^2r_0^2}\dot{\xi}_0 \tag{170}$$

and the velocity potential is therefore

$$\phi = \frac{r_0^2(1 - ikr_0)\dot{\xi}_0}{r(1 + k^2r_0^2)}e^{i[\omega t - k(r - r_0)]}. \tag{171}$$

From this we obtain at once the excess pressure at the surface of the pulsating sphere:

$$p(r_0) = \rho\dot{\phi}\Big|_{r = r_0} = \frac{\rho\omega r_0(kr_0 + i)\dot{\xi}_0}{(1 + k^2r_0^2)}e^{i\omega t}. \tag{172}$$

The radiation impedance of the device per unit area is evidently, since $\omega = kc$,

$$Z_R = \frac{p(r_0)}{\dot{\xi}_0 e^{i\omega t}} = \rho c\frac{(k^2r_0^2 + ikr_0)}{(1 + k^2r_0^2)}. \tag{173}$$

The first term in the numerator, a real quantity, is *in phase with the velocity* and is therefore *a resistance coefficient*; the imaginary quantity in the numerator is *in phase with the acceleration* ($i\omega\dot{\xi}_0$) and is therefore an *inertia coefficient.*

These coefficients may be written

$$\left.\begin{aligned} b_1 &= \rho c\frac{k^2r_0^2}{1 + k^2r_0^2} \quad \text{(resistance)},\\ a_1 &= \rho\frac{r_0}{1 + k^2r_0^2} \quad \text{(inertia)},\\ Z_R &= b_1 + i\omega a_1 \quad \text{(complex impedance)}. \end{aligned}\right\} \tag{174}$$

The form given is to be preferred, but the reader may express these in terms of wave length, by substituting $\frac{2\pi}{\lambda}$ for k. It will be noted at once that if r_0 becomes very small as compared with the wave length (i.e., if the generator becomes the point source of § 36) the radiation resistance becomes

$$b_1\Big|_{r_0 \doteq 0} = \rho c \cdot k^2 r^2{}_0 = \frac{\rho\omega^2 r_0{}^2}{c} \text{ per unit area.} \tag{174a}$$

This result enables us to liquidate a long-standing obligation, namely the determination of the radiation resistance of a small source of area S, assuming spherical divergence of sound energy therefrom. We have $r_0{}^2 = \frac{S}{4\pi}$ and consequently, *for the whole area*,

$$Sb_1 = b_1' = \rho c \frac{k^2 S^2}{4\pi} = \rho \frac{\omega^2 S^2}{4\pi c}, \tag{175}$$

which is the relation previously quoted.

We conclude with a discussion of the physical significance of the results obtained. Suppose first that the frequency of driving of the generator is low, i.e., λ is very great; we have already noted the form of b_1 for this case. The form of the *inertia term* a_1 shows that, in effect, a *mass of fluid* $r_0\rho$ per unit area of the generator surface must be pushed back and forth during the oscillation, much as if it were rigidly attached to the pulsating spherical surface of the generator. This is spoken of as the added mass due to the medium; in the case a *sound generator immersed in water* the added mass due to the water may be several times the mass coefficient of the generator structure itself, with the result that the natural frequency of the system is very much lowered. The added mass for the whole surface of the sphere is $4\pi r_0{}^3 \cdot \rho$; the total effect of added inertia due to the medium is thus equivalent to a quantity of the fluid *equal to that contained in a sphere three times as large* as the generator. This effect is analogous to that obtained for linear (oscillatory or non-oscillatory) motion of a sphere, at low velocities: in that case the added inertia is

that of *half* the quantity of fluid displaced by the sphere (Lamb, § 77).

On account of their inherent interest curves have been computed for the frequency variation of a_1 and b_1 for a pulsating sphere of unit radius, in fluid of unit density (see Fig. 13). (The *complementary* variation of a_1 and b_1 is reminiscent of similar relations observed previously, in pressure reactions as a function of frequency—it is interesting to compare the results shown in

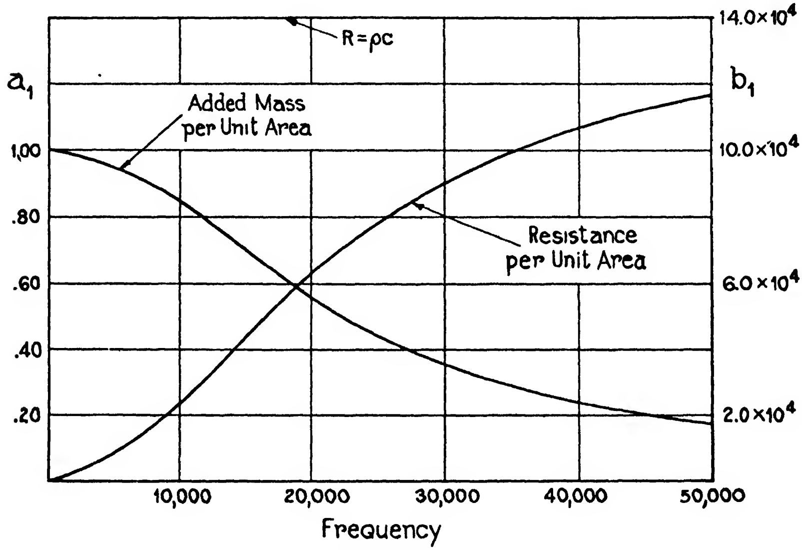

FIG. 13.—REACTIONS ON PULSATING SPHERE OF UNIT RADIUS IMMERSED IN WATER.

Fig. 5.) For the pulsating sphere, the change in a_1 and b_1 as higher frequencies are attained is striking; for very small values of λ, we have, since k is very great,

$$b_1 = \rho c = R, \qquad \text{cf. (117a)} \tag{176}$$

the characteristic *radiation resistance* of the medium previously obtained for plane waves. The *inertia term* (174) diminishes very rapidly with λ and is small if λ is less than (say) π in any practical case. The generator now has to work against only a *radiation resistance* (in addition to its own impedance) and its sound-generating efficiency is a maximum.

The results in this specific problem well illustrate the difficulties involved in "driving" an acoustic medium, i.e., in radiating sound without merely *pushing the medium away* at the driving point. These results are general in their import, and should be thoroughly understood by the reader before attacking other radiation problems. The remarks of Stokes (Lamb, § 80 or Rayleigh, II, § 324) are to the point in this connection; for his theory in full, reference may be had to his paper "On the Communication of Vibration from a Vibrating Body to a Surrounding Gas," in *Phil. Trans. Roy. Soc.*, 158, June 18, 1868, reprinted in his "Collected Papers," vol. IV, p. 299.

38. *Reactions of the Surrounding Medium on a Vibrating String*

The complete solution of this problem is a specialized and difficult matter, but it may be discussed in this place because it brings into one focus the important points relating to the interaction between vibrating system and medium that have been previously developed in the text. These points are, the radiation of energy from the system, the added mass, and finally the dissipation due to viscosity in the medium. And in connection with the last of these, there is an important application to the string galvanometer.

In the net result, the radiation from a vibrating string is the least important of the three effects, but it is necessary to deal with it first in order to get it out of the way. Anticipating the discussion of the next section (§ 40) it may be observed that any element of the string is virtually a double source, and therefore, except at very high frequencies, a radiator of low efficiency. Stokes, for example, in the classical memoir just cited pointed out how feebly the vibrations of the string are communicated to the surrounding gas. If lateral (i.e., non-radial) motion in the neighborhood of the string could be prevented, the intensity of the radiated sound would be enormously increased. "This shows the [vital] importance of sounding boards in stringed instruments. Although the amplitude of vibration of the particles of the sounding board is extremely small compared with that of

the particles of the string, yet as it presents a broad surface to the air it is able to excite loud sonorous vibrations, whereas were the string supported in an absolutely rigid manner, the vibrations which it could excite directly in the air would be so small as to be almost or altogether inaudible." (Stokes, quoted by Rayleigh, *loc. cit.*)

After a lengthy analysis, A. Kalähne (*Ann. d. Phys.*, 45, 1914, p. 657) finds that the radiation from an infinite string depends on the quantity

$$\kappa = \sqrt{k^2 - \left(\frac{2\pi}{\lambda'}\right)^2}, \tag{177}$$

in which $k = \dfrac{2\pi}{\lambda}$, for the medium (as is customary) and λ' is the wave length of the oscillations on the string. For steady state conditions in the (infinite) surrounding medium, it is found that for κ real (that is, if the wave velocity on the string is greater than the wave velocity in the medium) there is progressive wave motion, and hence radiation, from the string; but if κ is zero or imaginary, there are stationary waves throughout the medium, and no radiation takes place. In both cases the wave motion in the medium falls off as the distance from the string is increased; and even in the first case, the radiation is not very great. In view of these considerations and certain analytical difficulties, Kalähne has advisedly neglected the radiation resistance of the system in reckoning the added mass due to the medium.

Kalähne's work on the added mass (or lowering of the natural frequency of the system) is contained in two papers (*Ann. d. Phys.*, 45, 1914, p. 321, and *ibid.*, 46, 1915, p. 1) which deal with solid or hollow rods executing transverse vibrations in liquids or gases. The added mass due to the medium is not great unless the density of the medium is comparable to that of the rod (or string). The resultant lowering of frequency, for a solid string, is calculated to be

$$\frac{n - n'}{n} = \frac{L}{2} \cdot \frac{\rho}{\rho_0}, \tag{177a}$$

in which ρ is the mean density of the medium, ρ_0 the volume density of the string, n the frequency in vacuum, n' the actual frequency in the medium, and L a coefficient in terms of cylinder functions which is approximately unity in all practical cases, but which approaches zero for very high frequencies.

First we may remark that Kalähne's formula, at low frequencies (for which $L = 1$), is quite consistent with what we should expect on the basis of "equilibrium theory." It is well known that a cylinder moving sidewise in a fluid, at ordinary speeds, is burdened with an added mass equivalent to the quantity of fluid displaced by the cylinder. Consequently the added mass is to the inherent mass of a solid cylinder as $\rho : \rho_0$. Since the natural frequency of a stretched string is equal to a constant divided by the square root of the density of its material (§ 28), i.e., $n = \frac{C}{\sqrt{\rho_0}}$, we must have, for any increase $\delta\rho_0 \equiv \rho$ in the virtual density of the string, a negative increment δn of frequency which is equal to $\frac{n\rho}{2\rho_0}$.

To go deeper into the matter we may refer to Stokes' long memoir "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums," published in 1856.[1] Section III and the end of Section IV of this paper relate to a cylinder oscillating in a fluid in a direction normal to its axis. Stokes considered the viscosity of the fluid (which Kalähne did not) and was interested in the damping and the lengthening of the period of a cylindrical pendulum due the reactions of the surrounding air. Thus we should not expect his formulae to duplicate those of Kalähne as far as frequency variations in the reactions are concerned, but we should expect the low-frequency value of the added mass to be the same, and should obtain in addition a value of the damping due to friction.

Stokes' solution of the problem depends on a certain cylinder function $F(ma)$ [equations 85, 87, 93, of the original paper]

[1] *Camb. Philos. Soc. Trans.*, IX, 1856, p. 8: reprinted in Stokes' "Papers," vol. III, p. 1.

a being the radius of the cylinder while $m = \sqrt{\frac{in\rho}{\mu}}$. He obtained for the reaction on the cylinder, in terms of the acceleration $in\dot{\xi}$, the following expression:

$$M'\left(1 - \frac{4F_3'(a)}{m^2 a F_3(a)}\right)\cdot in\dot{\xi} \equiv M'(k - ik')in\dot{\xi}, \qquad \text{[Stokes, 98]}$$

in which $M' \equiv \pi a^2 \rho \cdot l$, the mass of fluid displaced by a length l of the cylinder. In this expression kM' is the added mass, while $nk'M'$ is the resistance coefficient; Stokes computed tables for k and k' for a wide range of values of the argument $\frac{1}{2}|m|a$.

For air, $\frac{\rho}{\mu} = 7.0$, approximately; consequently for a wire of radius .05 cm., vibrating at 1000 cycles, we should have $|m|a$ of magnitude about 10. In cases such as this we should expect to apply Stokes' solution for large values of the argument, namely,

$$k = 1 + \frac{2\sqrt{2}}{|m|a}, \quad k' = \frac{2\sqrt{2}}{|m|a} + \frac{2}{|m^2|a^2}. \qquad \text{[Stokes, end of Section III]}$$

From these we have, per unit length of cylinder, for the added mass,

$$a_1 = \pi a^2 \rho k = \pi a^2 \rho\left(1 + \frac{2}{a}\sqrt{\frac{2\mu}{n\rho}}\right); \tag{178}$$

and for the resistance coefficient,

$$b_1 = \pi a^2 \rho n k' = 2\pi a^2\left[\frac{1}{a}\sqrt{2n\mu\rho}\left(1 + \frac{1}{a}\sqrt{\frac{\mu}{2n\rho}}\right)\right]. \tag{178a}$$

It is evident that if $\mu = 0$, the added mass is exactly that of the equilibrium theory, and this value leads at once to the approximate formula already given for the lowering of the natural frequency. And considering only the effect of viscosity, this will be greatest at low frequencies, and will vanish asymptotically as the frequency is raised; but it must be noted that the formula cannot be correct at extremely low frequencies, because the development given depends on a value for $|m|a > 1$.

To find approximately the damping coefficient of the string we take the ratio of b_1 to twice the total (inherent and added)

mass of unit length of the string. The total mass is $\pi a^2(\rho_0 + \rho)$, to two terms. After a rough calculation we obtain

$$\Delta = \frac{b_1}{2\pi a^2(\rho_0 + \rho)} = \left(1 - \frac{\rho}{\rho_0}\right)\left(\frac{\sqrt{2n\rho\mu}}{\rho_0 a} + \frac{\mu}{\rho_0 a^2}\right), \quad (178b)$$

or, more simply still,

$$\Delta = \frac{\sqrt{2n\rho\mu}}{\rho_0 a}. \quad (178c)$$

We may now consider some recent experimental work. H. Martin (*Ann. d. Phys.*, 77, 1925, p. 627) made an experimental test of Kalähne's formula for the lowering of the natural frequency of steel wires when immersed in liquids. The frequency n is taken as unchanged, for the string immersed in air, on account of the smallness of the quantity $\frac{\rho}{2\rho_0}$; for the case of steel wires in water the formula is found to fit the facts very well, that is, to within one or two per cent.

Martin's apparatus consisted of a stretched wire, immersed in liquid and magnetically driven, the oscillations at various frequencies being observed with a microscope. In this way he got resonance curves, from which (if the vibrations were sufficiently small, to preclude over-driving) both the resonant frequency and the damping constant could be obtained. From a very careful study he found for steel wires the following relation between the damping constant of the system and the other quantities

$$\Delta = \frac{1.45\sqrt{\rho n\mu}}{a \cdot \rho_0}, \quad (178d)$$

which agrees closely with the formula we have just developed from Stokes' calculations.[1] We have, therefore, for certain

[1] Martin quotes a relation due to I. Klemencic (*Wien. Ber.*, 84, II Abt., 1881, p. 146), namely

$$\Delta = \frac{\sqrt{\rho n\mu}}{a\rho_0} + \frac{\mu}{\rho_0 a^2} - \frac{a^2\rho_0^2}{6\mu\rho},$$

for the damping constant of a cylinder immersed in a fluid, and oscillating about an axis perpendicular to its own. It is evident that the first term of the formula based on Stokes is in much better agreement with Martin's experiments than the first term of Klemencic's formula; the second term is common to both expressions.

cases, a fair theory of the damping of a vibrating string due to the viscosity of the surrounding fluid. Incidentally we note the close similarity between the damping constant as above given, and the attenuation factor α' (equation (R), Appendix A) for sound waves in a viscous medium surrounded by a tube; they differ merely by the factor $2c$ (twice the wave velocity)—a necessary matter of dimensions.

It would be interesting now if we could compute the added mass and damping of a *very fine* wire immersed in air. One would think that the "air damping" of a fine wire would be very great, and indeed it is; the Einthoven String Galvanometer is the outstanding application of this effect, as it depends for its success more on the fineness of the string than on any other factor. Stokes' theory, carried to its logical conclusion for small values of the argument $|m|a$ gives enormous values of added mass and resistance due to the viscosity of the medium (equation 115, original paper), but Stokes distrusted the application of the theory to this limiting case. Quoting from the end of Section IV of his paper, "It would seem that when the radius of the cylinder is very small, the motion of the cylinder would be unstable. This might well be the case with the fine wires used in supporting the spheres employed in pendulum experiments. If so, the quantity of fluid carried by the wire would be diminished, portions being continually left behind and forming eddies. The resistance to the wire would on the whole be increased, and would moreover approximate to a resistance which would be a function of the velocity." As one consequence of these speculations Stokes thought that the damping of a fine wire would in fact be greater than the value computed on the basis of the theory; and in view of this uncertainty we find it necessary to drop theoretical calculations at this point and deal with the case of a fine wire for the most part on an experimental basis.

The most recent representative paper on the Einthoven Galvanometer is that of H. B. Williams (*Jour. Opt. Soc. Am.*, 9, 1924, p. 129); it also contains references to other literature, including Einthoven's original contributions. Prof. Williams has in preparation a second paper dealing with his more recent ex-

perimental study of the galvanometer; meanwhile he has kindly made available this unpublished work, and his accumulated experience with the instrument, for the purposes of the discussion given here.

The substance of Williams' paper of 1924 is the theory of the motion of the current-carrying string, which is virtually a "uniformly loaded" member owing to the electrodynamic reaction between all the elements of the string and the perpendicular magnetic field in which it is placed. This theory of the action of the string under a uniformly distributed alternating force is of interest in itself, but that is not our present concern. It has been noted as an experimental fact that if a natural oscillation of one particular frequency is selected (harmonics being excluded) the damping factor is constant over a considerable range of amplitude (Williams, *loc. cit.*, p. 161). But in the theoretical studies of the galvanometer *per se*, no allowance has apparently been made for the variation of the damping constant with frequency; and what interests us at the moment is the validity of this idea.

Einthoven's experiments are recorded in a two-part paper in *Ann. d. Phys.*, 21, 1906, pp. 483–514 and pp. 665–700. He made measurements on certain strings under widely varied adjustment of the tension; of these we shall consider two typical cases, namely, the high-tension or oscillatory case, and the low-tension or over-damped case, for which n_0 was less than Δ. For the oscillatory case, Einthoven measured the logarithmic decrement (δ) and determined the mechanical resistance r from the following relations:

$$\Delta = \frac{\delta}{T}, \qquad r = 2m_0\Delta, \qquad m_0 = \frac{s}{n_0^2},$$

in which T is the *actual* period of the oscillation, and $\frac{n_0}{2\pi}$ the natural frequency. It is evident in this case that m_0 is an *ideal or derived mass coefficient*, which is valid for the case of rapid oscillations, but which may not be applicable to the slower motion of the string under reduced tension.

Now consider the over-damped case. Einthoven made

a number of measurements of the mechanical resistance (r) of the system which were based on the *point of inflection* of the curve of the growth of the deflection when a constant force (current) was applied. Now (as has been hinted in problem 2, Chap. I), at the point of inflection, the acceleration is zero, and we have

$$q = \frac{r}{s}\dot{\xi}, \qquad \left(q = \frac{F}{s} - \xi; \qquad \dot{q} = -\dot{\xi}\right).$$

Hence *at this point* r is given in terms of the stiffness constant (s) and the graphically measured quantities q and $\dot{\xi}$, but it must be remembered that these relations depend on unvarying mass and resistance constants, which Einthoven's analysis assumes. It is safe to assume that the stiffness is constant, in a given adjustment; but if mass and resistance vary, we should prefer to write, for the relation between displacement and velocity at the point of no acceleration,

$$q = \frac{2\Delta_i}{n_0^2}\dot{\xi},$$

in which the subscript i refers to conditions at the point of inflection. This follows from the equation (obtained by putting $\ddot{\xi} = 0$ in the original differential equation),

$$2\Delta_i\dot{\xi} + n_0^2\xi = 0, \qquad \left(\Delta_i = \frac{r_i}{2m_i}\right)$$

which we should be compelled to use if the stiffness were taken as constant in the experiments, and the variations in r, m and Δ were unknown.

Now Einthoven apparently thought (*loc. cit.*, p. 503) that r was nearly constant, because when measured in the manner described above, the value of r was nearly the same for a wide variation in the tension of the string—in one test the tension varied over a range in the ratio of 1 : 324. But from the situation as we have examined it, it would seem more reasonable to give the final results in terms of Δ rather than r, because of the possibility that variations in r and m might escape notice,

as long as Δ remained a nearly constant quantity. It is likely that both the added mass and the resistance of the string vary; and we should expect them to vary in the way predicted by Stokes' theory.

Indeed Einthoven (*loc. cit.*, p. 670) has noted that *under reduced tension* the string behaves as if there were a variation in mass—his measurements being made on the assumption of a constant resistance. (He found the mass to be greater, the slower the motion—which would be in harmony with the theory.) We may now observe that it is entirely possible, within a certain range of frequencies, for both mass and resistance to vary in such a way that one compensates for the other, and a more or less constant value of damping results. One of Williams' tests may be mentioned in this connection. The damping and natural frequency (1200 cycles) of a galvanometer string were first measured, for a certain tension; it was then calculated, on the assumption of constant damping, that the string would be aperiodic at 400 cycles. The tension was then correspondingly reduced, to give this natural frequency, and the string was found to be aperiodic, to within one-third of one per cent of the steady current deflection. It seems reasonable to conclude that when the motion of the fine string is very slow, there must be a certain compensating effect between the variations in added mass and in resistance. Thus from the experimental standpoint, we have in the fine air-damped string a most curious sequel to Stokes' theory, particularly when we note the theoretical difficulties which he conceived, and his scruples in dealing with them. Some rough computations on the basis of Stokes' theory for a very fine wire give values of added mass and resistance which are not incompatible with the experimental phenomena we have described, and the reader who wishes to go further into the matter should examine the behavior of Stokes' functions k and k' for very small values of the argument (ma).

A word may finally be said on electrical damping. Any galvanometer of low electrical resistance may be strongly damped by suitably adjusting the external circuit. If the instrument in itself has very variable mechanical characteristics, it is often of

advantage to stabilize it in this way. In the case of the Einthoven Galvanometer the fineness of the string, to which the sensitivity and the damping of the instrument are due, entails a high electrical resistance, so that electrical damping is hard to apply; but as we have seen, it is not required. There is, however, a very nice theoretical point in this connection; it will be found in a paper by L. S. Ornstein (Kon. Akad. Amsterdam, Proc., XVII, Nov. 28, 1914, p. 784) on the Theory of the String Galvanometer. It is that electrical damping only occurs for those frequencies which are odd multiples of the fundamental frequency of the string—and, since only the odd segment of the string contributes damping, the most important cases are for the vibrations of the lower frequencies. Ornstein's paper has other interesting features, and it appears to be the most compact treatment available of the theory of this instrument.

PROBLEMS

21. A plane sound wave in air has an intensity of 1 erg per second. What is the force on 10 sq. cm. of an infinite wall (normal to the wave), due to the impact of the wave?

22. Suppose in problem 21 the area considered in the wall is a piston whose constants are $b = 200$ c.g.s., $c = 10^7$ c.g.s. and $a = 1$ gram. What is the steady state equation of motion of the piston?

23. What is the added resistance (if any) of the piston system due to the fact that it is radiating into a *semi-infinite* medium?

24. A plane wave of sound in water strikes a plane water-air boundary at an angle of 45°. Investigate the properties of the reflected and transmitted waves. How are these changed if the rare and dense media are interchanged with respect to the boundary?

25. In a wave power transmission line the diameter of the tube is 3 cm. and the fluid used is water; the line is operated at 100 cycles. Assuming Helmholtz's law of resistance in the tube [Appendix A, eq. (R)] what is your engineering opinion as to the maximum distance for economic power transmission, and on what criteria are your conclusions based? How would your results be modified if oil of viscosity $\mu = 1.0$ and density $\rho = 0.9$ were used in place of water?

26. Fluid of viscosity μ is contained in a narrow crevice bounded by parallel walls separated by distance d. Applying a method similar to that of Appendix A, show that the mean resistance coefficient is $R = \frac{12\mu}{d^2}$ as given in equation (37), Chap. I.

27. Pistons with elastic constraints, but free from resistance, are disposed at either end of a tube of length l, containing air, as in theory of § 32. Neglecting dissipation in the tube, show that the natural frequencies of the system are given by the equation

$$\tan \beta l = \frac{-\,i(Z_0 Z_l + \rho^2 c^2)}{\rho c(Z_0 + Z_l)}.$$

On the basis of this equation discuss the more interesting cases of resonance of the system, for various values of l, and various relations between Z_0, Z_l, and ρc.

28. Taking the velocity potential as

$$\phi = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right) \cos \omega t$$

find the natural frequencies of a tube of length l, rigidly closed at both ends (§ 33). Find also the space-distribution of amplitude and pressure in the tube in the normal modes.

29. A pulsating sphere of radius 10 cm. is tuned to 1000 cycles and has an inherent (mechanical) resistance of 20 c.g.s. per unit area at that frequency. Assuming a force $\Psi_0 e^{i\omega t}$ per unit area applied to its surface, find the ratio of radiated energy to total energy expended, at 1000 cycles, when immersed (1) in hydrogen, (2) in air, and (3) in water. What do you find for the relative sound intensities at corresponding points in the field, in the three media?

30. Two similar sirens are working, one in air, near the ground, in flat country; the other just below the surface of the water, in a quiet sea. What are their relative rates of working, in order that a listener at a given distance may perceive sound of the same intensity, assuming equally efficient listening devices in the two media and neglecting all degrading factors? Now assuming dissipation due to viscosity in the two media, how are your conclusions modified for components of the siren tone of 1000 and 3000 cycles?

CHAPTER IV

Radiation and Transmission Problems

40. *General Considerations; Single and Double Sources*

In the preceding chapters the emphasis has been placed on methods rather than results, and as a consequence it has been possible to develop the general theory with some degree of unity and coherence. In the present chapter departure to some extent from this plan is unavoidable, for we shall have to consider several applications in detail in order to give a fair idea of the range of usefulness of the theory. In addition, there are serious analytical difficulties, and it will be necessary at times to depart from a strictly rigorous procedure in order to reach practical conclusions without undue labor.

Before turning to typical radiation problems we may briefly mention one of the ingenious conceptions of the classical theory, namely *the double source*, or *acoustic doublet*. This is constituted of two small simple sources, close together, and identical except for the sign of the strength factor A; the velocities of emission of fluid at the two sources are 180° out of phase. The potential theory of the acoustic doublet is analogous to that of the magnetic doublet, and of the electrostatic doublet; the idea originated from the necessity of determining the acoustic field due to such devices as tuning forks, membranes with both sides exposed to the medium, etc., which exert equal and opposite driving forces on the medium at two nearby points.

Let $\pm A'$ be the strength of either component of the double source, and δx be the distance between components, the axis of the doublet being parallel to OX. It may then be shown that the velocity potential at a point distant r from the doublet is

$$\phi = -\frac{A'\delta x}{4\pi}\frac{\partial}{\partial x}\left(\frac{e^{i\omega\left(t-\frac{r}{c}\right)}}{r}\right) = -\frac{A}{4\pi}\frac{\partial}{\partial r}\left(\frac{e^{i\omega\left(t-\frac{r}{c}\right)}}{r}\right)\cos\theta, \quad (179)$$

in which $A = A'\delta x$, *a finite quantity*, and $\delta r = \delta x \cos\theta$, θ being the inclination of the radius vector r to the axis of the doublet. Performing the differentiation we have, if $k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$,

$$\phi = \frac{(1 + ikr)A}{4\pi r^2} \cos\theta \cdot e^{-ikr} e^{i\omega t}. \tag{179a}$$

When kr is small, i.e., for points near the doublet, the velocity potential varies as $\frac{1}{r^2}\cos\theta$; for large values of r, the corresponding factor in the coefficient is $\frac{ik}{r}\cos\theta$. Thus along the axis, at great distances, conditions are not very different from those prevailing in the field due to a single source. For $\theta = \frac{\pi}{2}$ (i.e., when r is normal to the axis) the radiation from the doublet is zero; and it is evident that on the average (taking only energy into account) the doublet is a poorer radiating device than the simple source of equivalent strength.

According to present practice in acoustics we deal with sound-generating and detecting devices in which only *one side* of the vibrating element or diaphragm is exposed to the medium; the notion of a double source is not of great use in these problems and we shall not pursue it further. It is of greatest advantage in the solution of such problems as the radiation from a *vibrating* sphere (Lamb, § 77; Rayleigh, II, § 328) or the scattering of sound waves by a spherical obstacle (Lamb, § 81); the reader interested in such matters may refer to the classical treatises.

A major problem which we shall consider is the radiation from a circular piston. To avoid analytical complications the piston is made to reciprocate in a cylindrical hole in a plane rigid wall; the radiation thus takes place into a semi-infinite medium. The effect of the piston on the medium can be considered as the sum of the effects of elementary sources, each of strength $dS \cdot \dot{\xi}_0$, distributed uniformly over the area S of the

piston. Since the divergence of the radiation from each elementary source is limited by the wall to a solid angle 2π, instead of the angle 4π of eq. (160), we have, for the potential at any point distant r from the elementary source

$$\phi = \frac{dS \cdot \dot{\xi}_0}{2\pi r} e^{i\omega t} e^{-ikr}, \tag{180}$$

and consequently the velocity potential due to the piston as a whole is

$$\phi = \frac{\dot{\xi}_0{}^{i\omega t}}{2\pi} \iint_S \frac{e^{-ikr}}{r} dS. \tag{180a}$$

This useful formula (the existence of which was implied in the discussion following eq. (160) has been obtained by simple reasoning on physical grounds; but it is a particular case of a general formula which can be established rigorously. The general formula (Rayleigh II, § 278, § 302) is

$$\phi = -\frac{e^{i\omega t}}{2\pi} \iint_S \frac{\partial\phi}{\partial n} \frac{e^{-ikr}}{r} dS, \tag{180b}$$

in which $-\frac{\partial\phi}{\partial n}$ is the normal velocity of the element dS of the reciprocating surface; in (180) $-\frac{\partial\phi}{\partial n} = \dot{\xi}_0$ and, being a constant with respect to dS, is removed from the integrand. Equation (180a) or (180b) can be taken as the starting point for the investigation of radiation into a semi-infinite medium from the vibrating surface of any solid body, providing $-\frac{\partial\phi}{\partial n}$ is known for each point of the surface.

41. *High-Frequency Radiation from a Piston; Diffraction*

From the principles established in § 37 and § 40 we can proceed to investigate the radiation from a flat piston into the semi-infinite medium, the piston being driven at any frequency whatever. Before dealing with this question generally, which leads to some detailed applications, we had best dispose of the

particular case of radiation at high frequencies, with its accompanying diffraction phenomena.

Taking the piston of large diameter as compared with the wave length, it may safely be assumed that the waves will be nearly plane for a little distance from the piston, and consequently there will be no added mass due to the medium. Moreover, the radiation resistance will be that for plane waves, i.e., $R_1 = \rho c$ per unit area of the piston. At some distance from the piston, however, the radiation will begin to diverge; thereafter the intensity must weaken and ultimately become inversely proportional to the square of the distance from the source. These conclusions are all analytically verified, in this and the succeeding article.

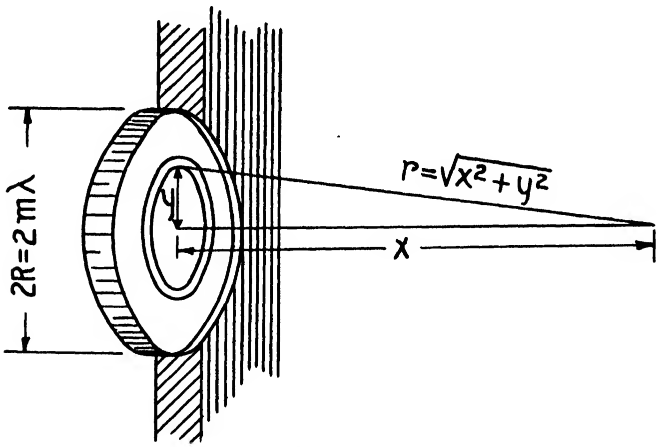

FIG. 14.

It will suffice to investigate conditions on the axis of symmetry of the system, in which case the integration of equation (180*a*) presents no difficulty. If the piston is given a motion $\xi_0 \cos \omega t$, the velocity potential is, in the notation of Fig. 14,

$$\phi(x) = \frac{\dot{\xi}_0 e^{i\omega t}}{2\pi}\int_{y=0}^{y=R} \frac{2\pi y\,dy}{\sqrt{x^2+y^2}} e^{-ik\sqrt{x^2+y^2}} = \frac{i\dot{\xi}_0}{k} e^{i\omega t} e^{-ik\sqrt{x^2+y^2}}\Bigg|_0^{y=R}, \tag{181}$$

in which it is understood that only the real component is to be

retained. Suppose now that $R = m\lambda$; inserting the limiting values of y, and reverting to real quantities,

$$\phi = -\frac{\dot{\xi}_0}{k}[\sin(\omega t - k\sqrt{x^2 + m^2\lambda^2}) - \sin(\omega t - kx)], \qquad (182)$$

which is rigorously correct for all points on the axis. Since $k = \frac{2\pi}{\lambda}$ it is seen that for $x = 0$, the maximum value of ϕ at the center of the piston face is zero or $\frac{2\dot{\xi}_0}{k}$, according as $m = n$, or $m = n + \frac{1}{2}$, n being an integer. It is also evident (on differentiating with respect to x) that the velocity at $x = 0$ is $\dot{\xi}_0 \cos \omega t$, as it should be.

To investigate conditions at points along the axis for which $x > 3m\lambda$, we are justified in using the approximation

$$\sqrt{x^2 + m^2\lambda^2} = x\left(1 + \frac{m^2\lambda^2}{2x^2}\right) = x + \alpha, \qquad (183)$$

and, since

$$\sin(\theta - k\alpha) - \sin\theta = -2\cos\left(\theta - \frac{k\alpha}{2}\right)\sin\frac{k\alpha}{2},$$

we have

$$\phi = +\frac{2\dot{\xi}_0}{k}\sin\frac{k\alpha}{2}\cos\left(\omega t - kx - \frac{k\alpha}{2}\right). \qquad (184)$$

The maximum value of the pressure is

$$p = \rho\dot{\phi}_{\text{max.}} = \frac{2\rho\dot{\xi}_0\omega}{k}\sin\frac{k\alpha}{2} = 2\rho c\dot{\xi}_0\sin\frac{\pi}{2}\cdot\frac{m^2\lambda}{x}, \qquad (185)$$

as $k = \frac{2\pi}{\lambda} = \frac{\omega}{c}$; the intensity is therefore, by (164*a*)

$$\frac{dW}{dt} = \frac{1}{2}\frac{p^2_{\text{max.}}}{\rho c} = 2\rho c\dot{\xi}_0^2\sin^2\left(\frac{\pi}{2}\cdot\frac{m^2\lambda}{x}\right). \qquad (186)$$

From (186) we infer that there is an approximately parallel beam of radiation out to the point $x = m^2\lambda$; that along the axis, *within the region* $0 < x < m^2\lambda$ there is a succession of maxima and minima of intensity (analogous to the bright and

dark spots of optical diffraction phenomena) due to the interference of rays from the different portions of the piston; and finally that for large values of x the intensity is, for points on the axis,

$$\frac{dW}{dt} = \frac{\pi\rho c}{2}\frac{m^2}{x^2}\cdot\pi R^2\dot{\xi}_0^2. \tag{186a}$$

It is evident that the radiation *tends* to diverge, ultimately in the form of a cone, and that after the critical distance $x = m^2\lambda$ has been exceeded, this divergence becomes a dominating factor. Equation (186a) can be written in the form of (164) and this leads to an interesting but somewhat paradoxical result. Noting that $\pi R^2 = \pi\lambda^2 m^2 = S$, and $\lambda = \dfrac{2\pi c}{\omega}$, we have from (186$a$)

$$\frac{dW}{dt} = \rho c\frac{k^2 S^2\dot{\xi}_0^2}{8\pi^2 x^2}. \tag{186b}$$

It thus appears, on comparison with (164) that at great distances, using the axial radiation from a large piston source, a fourfold gain in intensity at a given point can be obtained as compared with that due to a small spherical source of equal strength. It will be noted that aside from this numerical factor there is no special factor in (186b) to differentiate it from (164); that is, the particular angle subtended by the cone of rays has no direct effect on the intensity along the *axis* of the cone. *This is not to say, however,* that the solid angle containing the cone of rays is indeterminate, or independent of the dimensions of the particular piston used. A rough measure of this solid angle is the angle subtended by the piston when viewed from the point $x = m^2\lambda$; this is

$$\Omega = \frac{\pi\lambda^2 m^2}{m^4\lambda^2} = \frac{\pi}{m^2}. \tag{187}$$

Evidently, for a given wave length, the divergence of the conical beam varies inversely as the area of the piston.

For similar relations as between wave length and the size of the source, the acoustic diffraction phenomena are exact analogues of optical diffraction effects. The comparison is of interest in the present case, if we regard the piston as analogous to a circular aperture in a plane screen against which light waves are normally incident. In the solution of this corresponding optical problem, the Fraunhofer point of view may be adopted; this assumes parallel rays in both the incident and diffracted beam. The *angle of diffraction* for the first *minimum* of intensity is $.61\frac{\lambda}{R}$, if R is the radius of the aperture; this may be taken, without serious error, at moderate distances from R (e.g., for $x > 5R$) as the angle subtended by the limiting radius of the *first bright diffraction circle* on a screen normal to the axis when viewed from the aperture. The point we are leading up to is that as long as first bright diffraction circle is *smaller* than the size of the aperture, we have substantially a parallel beam, and that, as the *distance* from the aperture is increased, the first bright diffraction circle expands proportionately, while divergence of the beam into nearly conical form takes place. The solid angle of the cone thus asymptotically filled by this important component of the radiation is

$$\frac{\pi(.61\lambda)^2}{R^2} = (.61)^2 \cdot \frac{\pi}{m^2}. \tag{187a}$$

This may be taken as a more accurate expression for Ω than that given in (187). To summarize, there is evidently some point on the axis in the neighborhood of $x = m^2\lambda$ which roughly marks the beginning of the transition from a parallel to a diverging beam of radiation. The hydrodynamical method of studying diffraction problems is of great advantage in many cases, and *rigorous solutions* can be obtained providing we avoid such approximations as have been made in the present example.

The example we have treated finds some application in *high-frequency* submarine signalling. Low-frequency submarine signalling is well treated in Drysdale (Chap. IX); but there are

few references [1] to high-frequency work in technical literature, and a brief reference to the practical side of such operations may therefore be permissible. The economy of high-frequency signalling depends not only on the concentration of the radiation (from the energy standpoint) but also on the associated directive property. Obstacles, such as ships, icebergs, rocky reefs, etc., can be located at distance of a mile or more by the "echo" method. The best high-frequency sound generator known is a sandwich-like tuned structure made of sections of quartz crystals, operated piezo-electrically, which are firmly mounted between two iron slabs each a quarter wave length thick. One side of the vibrator is of course shielded from the medium. When operated at (say) 50,000 cycles, by being placed in connection with a source of high-frequency electrical energy, the vibrator behaves as a condenser with a certain amount of internal ohmic resistance; as much as one-third of this may be due to its radiation resistance as a sound generator. This radiating efficiency is considerable, being limited only by such dielectric and elastic hysteresis effects in the structure as are unavoidable with the substances used. For detection of high-frequency sound waves a structure similar to that of the vibrator may be used, or the vibrator itself as in echo work, taking advantage of a quick change from sending to receiving circuits through a multiple key. The area of the vibrator may be of the order of 400 sq. cm.; it thus intercepts considerable energy when used as a detector. Carbon microphones, and piezo-electric crystals of rochelle salt may also be used as detecors if suitably mounted in resonant structures. The electrical circuits required for sending and receiving are similar to those used in radio transmission.

[1] The sketch following is of certain unpublished experiments made by the Columbia University Group (Profs. M. I. Pupin, A. P. Wills, and J. H. Morecroft) in 1918. The reader may compare a similar sketch of the Experiments of Langevin and Chilowsky given in *Nature*, May 9, 1925, pp. 689–90, in the article "Echo Sounding"; see also French Patent No. 505, 703 (1918) and British Patent No. 145,691 (1920) to Langevin, for the Quartz Piezo-Electric vibrator. A more complete description of Langevin's apparatus is given in "Ultra Sonic Waves for Echo Sounding," *Hydrographic Review*, (*Monaco*) II, No. 1, 1924, p. 57.

The frequency used in underwater high-frequency work should be less than 100,000 cycles, owing to various practical considerations. One of these is of course the kinematic viscosity factor, which accounts for appreciable attenuation at very high frequencies; other limiting factors are the difficulties encountered in the design of apparatus, and in utilizing available materials. Temperature and current conditions in the water are also disturbing factors which are aggravated at high frequencies; these are mentioned in § 34.

42. *Radiation from a Piston into a Semi-Infinite Medium*

We return to the discussion of the general problem of radiation from the vibrating piston at frequencies for which divergence near the piston must be taken into account. If we can determine the reaction on the piston as a function of frequency, we shall have material useful in the solution of several other problems.

We start again from the equation

$$\phi = \frac{\dot{\xi}_0 e^{i\omega t}}{2\pi}\int\!\!\int_S \frac{e^{-ikr}}{r}\,dS, \tag{180a}$$

the geometrical relations being, for the moment, as in Fig. 14. The pressure at any point y on the piston is therefore

$$p(y) = \rho\dot{\phi}\Big|_{r=y} = \frac{i\rho\omega\dot{\xi}_0 e^{i\omega t}}{2\pi}\int\!\!\int_S \frac{dS}{r_1}e^{-ikr_1}, \tag{188}$$

in which r_1 denotes a radius vector lying in the surface of the piston and extending from the point y to the surface element dS. Thus to compute the force on the piston we must perform the integration in (188) and then integrate the resulting pressure (a function of y) over the surface of the piston. The force is then

$$\Psi = \int\!\!\int_S p\,dS' = \frac{i\omega\rho\dot{\xi}_0 e^{i\omega t}}{2\pi}\int\!\!\int_S dS'\int\!\!\int_S \frac{dS}{r_1}e^{-ikr_1}, \tag{189}$$

in which dS' denotes a surface element at the point y. The situation is as shown in Fig. 15: each element dS is to be summed with respect to each element dS' over the surface of the disc; the effect at dS due to dS' is the same as the effect at dS' due to dS, so that in the double summation, the ultimate result will be *twice as great* as if each pair of elements dS, dS' were taken only *once*. The problem can thus be simplified if we *multiply* the quantities to be integrated *by two*, and then

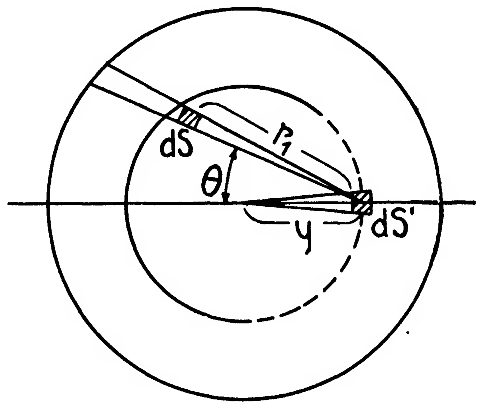

Fig. 15.

arrange the integration so that each pair of elements is taken *only once*. This latter end is accomplished if for each value of y (i.e., for each dS') we integrate with respect to dS only over the inner circular portion of the disc; that is, the portion of radius y, thus keeping dS' outside the region of the first integration.

We have first to evaluate the integral

$$V = \iint_S \frac{dS}{r_1} e^{-ikr_1} = \iint \frac{r_1 dr_1 d\theta}{r_1} e^{-ikr_1} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2y\cos\theta} e^{-ikr_1} dr_1 d\theta. \tag{190}$$

The calculation is treated by Rayleigh (II, § 302) but the fol-

lowing outline may be useful. We can take, instead of (190) the integral

$$2\int_0^{\frac{\pi}{2}}\int_0^{2y\cos\theta} e^{-ikr_1}dr_1d\theta = -\frac{2}{ik}\int_0^{\frac{\pi}{2}}(e^{-i2ky\cos\theta} - 1)d\theta,$$

that is

$$V = \frac{\pi}{ik} + \frac{2i}{k}\int_0^{\frac{\pi}{2}} e^{-i2ky\cos\theta}\,d\theta. \tag{190a}$$

Expanding the exponential function, and integrating term by term, we have

$$\int_0^{\frac{\pi}{2}} e^{-i2ky\cos\theta}d\theta = \frac{\pi}{2}\left[1 - \frac{1}{2^2}(2ky)^2 + \frac{1\cdot 3}{2\cdot 4\cdot 4!}(2ky)^4\ldots\right]$$
$$- i\left[2ky - \frac{2}{3\cdot 3!}(2ky)^3 + \ldots\right], \tag{190b}$$

since

$$\int_0^{\frac{\pi}{2}}\cos^n x dx = \frac{1\cdot 3\ \ldots\ (n-1)}{2\cdot 4\ \ldots\ n}\frac{\pi}{2} \quad \text{if } n \text{ is even,}$$

and

$$\int_0^{\frac{\pi}{2}}\cos^n x dx = \frac{2\cdot 4\ \ldots\ (n-1)}{1\cdot 3\ \ldots\ n} \quad \text{if } n \text{ is odd.}$$

The first parenthesis in (190*b*) will be identified by the reader as $J_0(2ky)$; the second parenthesis is a related but odd function which Rayleigh defines as

$$\frac{\pi}{2}\cdot K(z) = \left(z - \frac{z^3}{1^2.3^2} + \frac{z^5}{1^2.3^2.5^2} - \ldots\right). \tag{190c}$$

It must be noted that this is not a *complete* Bessel's function of the second kind, since by itself it does not satisfy Bessel's equation.

Equation (190*b*) now becomes

$$\int_0^{\frac{\pi}{2}} e^{-i2ky\cos\theta}d\theta = \frac{\pi}{2}[J_0(2ky) - iK(2ky)]; \tag{190d}$$

placing this in (190*a*) we have for the final result

$$V = \frac{\pi}{k}[K(2ky) - i(1 - J_0(2ky))]. \tag{191}$$

V may be considered the Newtonian Potential at the edge of a disc of radius y, the density of the disc being e^{-ikr_1}, if r_1 is taken with respect to the point on the edge.

In performing the second integration, namely $\int^{S} V \cdot dS'$ we note that $dS' = 2\pi y dy$; consequently we shall need the following formulae for the integration of the Bessel's functions:

$$\left.\begin{aligned} &\int_0^b J_0(z)\cdot z dz = zJ_1(z)\Big|_0^b, \\ \text{and} \quad &\int_0^b K(z)\cdot z dz = K_1(z)\Big|_0^b. \end{aligned}\right\} \tag{192}$$

The first formula is that which follows eq. (47) of § 12; the second formula results from the definition of $K(z)$, if $K_1(z)$ is as given below in (195).

Using these relations, and taking *twice* the values obtained from the integrations, we have, for the force on the piston

$$\Psi = \frac{i\omega\rho\xi_0 e^{i\omega t}}{\pi}\int_0^R 2\pi y dy\cdot\frac{\pi}{k}[K(2ky) - i(1 - J_0(2ky))]$$

$$= i\omega\rho\pi\xi_0 e^{i\omega t}\left[\frac{1}{2k^3}K_1(2kR) - i\frac{R^2}{k}\left(1 - \frac{J_1(2kR)}{kR}\right)\right], \tag{193}$$

or, since $\omega = kc$,

$$\frac{\Psi}{\xi_0 e^{i\omega t}} = \pi R^2\cdot\rho c\cdot\left(1 - \frac{J_1(2kR)}{kR}\right) + i\frac{\omega\rho\pi}{2k^3}K_1(2kR). \tag{193a}$$

This expression is in the standard form for an impedance, $Z = S(b_1 + i\omega a_1)$; in the first term (i.e., in b_1) we should expect $\frac{J_1(2kR)}{kR}$ to vanish when kR is large (and it does), thus making the radiation resistance of the piston $\pi R^2\cdot\rho c$ at high frequen-

cies.[1] (This is consistent with the theory of § 41). The general behavior and the applicability of the functions $J_1(2kR)$ and $K_1(2kR)$ are fully dealt with by Rayleigh (*loc. cit.*); these are matters we shall take for granted. For our purposes it suffices to note a few terms of the series

$$\left.\begin{aligned} J_1(z) &= \frac{z}{2}\left[1 - \frac{z^2}{2.2^2} + \frac{z^4}{2.4.2^2.6} - \frac{z^6}{2.4.6.2^2.6.8} + \dots\right], \\ 1 - \frac{J_1(2kR)}{kR} &= \frac{k^2R^2}{2} - \frac{k^4R^4}{2^2.3} + \frac{k^6R^6}{2^2.3^2.4} - \frac{k^8R^8}{2^2.3^2.4^2.5} + \dots; \end{aligned}\right\} \quad (194)$$

and

$$K_1(z) = \frac{2}{\pi}\left(\frac{z^3}{3} - \frac{z^5}{3^2.5} + \frac{z^7}{3^2.5^2.7} - \dots\right). \quad (195)$$

In treatises on Bessel's functions it is shown that for *large* values of the argument

$$K_1(z)\Big|_{z \text{ large}} = \frac{2z}{\pi}. \quad (195a)$$

We are now prepared to discuss the reaction of the medium on the piston at *low* frequencies, which is the more important case, in general acoustics.

Since kR is small, we need take only the first term in the lower series, equations (194); this gives for the resistance coefficient

$$Sb_1 = \pi R^2 \cdot \rho c \cdot \frac{k^2R^2}{2} = \frac{\rho c \cdot k^2(\pi R^2)^2}{2\pi} = \frac{\rho\omega^2 S^2}{2\pi c}. \quad (196)$$

It is interesting to compare this with what we should have obtained for *half* the pulsating sphere of § 37, assuming it to radiate into a *semi-infinite medium.* In that case, taking $S = 2\pi r_0^2$ we should have had, instead of (175)

$$Sb_1 = \rho c \cdot \frac{k^2 S^2}{2\pi} = \frac{\rho\omega^2 S^2}{2\pi c}. \quad (175a)$$

[1] The resistance and reactance factors in the impedance of the piston are plotted as X and Y (functions of kR) in Fig. 19 of § 46.

Thus, as far as the radiation *resistance* is concerned the piston is equivalent to the pulsating hemisphere of equal area, at low frequencies. At high frequencies the resistance is ρc per unit area in either case.

However, at low frequencies the added mass factor due to the medium is slightly greater for the piston of equal area, than for the pulsating hemisphere. For the latter we have found, in § 37,

$$Sa_1 = 2\pi r_0{}^2 \cdot r_0 \rho = \rho \frac{S^{3/2}}{\sqrt{2\pi}}. \tag{197}$$

For the piston we have, at low frequencies, from (193*a*) and (195)

$$i\omega a_1 S = \frac{i\omega\rho\pi}{2k^3} \cdot \frac{2(2kR)^3}{3\pi} = i\omega\rho \cdot \frac{8}{3}R^3, \tag{198}$$

that is,

$$Sa_1 = \pi R^2 \cdot \frac{8R}{3\pi} \cdot \rho = \rho \cdot \frac{8}{3}\left(\frac{S}{\pi}\right)^{3/2}. \tag{198a}$$

The numerical coefficients in (197) and (198*a*) are in the ratio 0.40 : 0.48, approximately; thus, at low frequencies, the added mass due to the semi-infinite medium is 20 per cent *greater for the flat disc* than for the hemisphere of equal area.

The added mass becomes vanishingly small at high frequencies. Using the relation (195*a*) and the imaginary term of (193*a*) again, we have, for the piston

$$i\omega a_1 S = i\frac{\omega\rho\pi}{2k^3} \cdot \frac{4kR}{\pi},$$

that is, since $k = \dfrac{2\pi}{\lambda}$,

$$Sa_1 = \frac{2\rho}{k^2}R = \frac{\rho\lambda^2}{2\pi^2}\sqrt{\frac{S}{\pi}}. \tag{199}$$

The corresponding factor for the pulsating hemisphere is, from (174), since λ is small,

$$Sa_1 = \frac{\rho\lambda^2}{2\pi}r_0 = \frac{\rho\lambda^2}{2\pi}\sqrt{\frac{S}{2\pi}}. \tag{174b}$$

The reader may easily convince himself that the inertia coefficients, in either case, are negligible at high frequencies, as compared with the radiation resistance $Sb_1 = \rho cS$.

The utility of the results obtained for the piston will appear presently. An immediate application is involved in determining the impedance of the *semi-infinite medium* at the end of a tube of circular cross-section which is fitted with a *plane flange extending to infinity*. Within the tube just before the end is reached, the waves are plane; at the opening we can assume (with the chance of a very small error) that they diverge into the medium as if produced by a piston oscillating in the end of the tube. Now abolishing the piston, it is evident that, if the motion is to continue, a force must be applied at the circular opening which is equal to the product of the velocity at the opening and the impedance we have determined, namely $S(b_1 + i\omega a_1)$, which represents the reaction of the medium on the piston. The error involved in assuming that the distribution of velocity in the opening is the same as if the piston were present is a small quantity of the second order.

43. *End Corrections for a Tube; Impedance of a Circular Orifice*

Before making the application suggested above, to "correct" the theory of the finite open ended tube for the impedance of the medium at the end, it may be well to inquire just what such a correction implies. Damping (due to radiation) and inertia (due to the divergence of the wave motion) are added at the end; considering the tube as a simple vibrating system, it is clear that a considerable increase in dissipation is required to affect the frequency of the natural oscillations, while any accession of inertia is directly effective in lowering the natural frequency. Inasmuch as the natural frequencies of a finite tube are inversely proportional to its length we should expect the principal effect of the terminal impedance (i.e., the added inertia) to result in a slight increase in the effective length of the tube. The same conclusion is reached if the argument is based on dimensional grounds.

We know from the tube theory of § 33 that the particle velocity for a tube open at $x = 0$ and closed at $x = l$ is

$$\dot{\xi}(x) = \dot{\xi}_0\left(\cos\frac{\omega x}{c}\right)e^{i\omega t} \quad \text{if} \quad \cos\frac{\omega l}{c} = 0, \text{ (cf. 137, 138)},$$

$\dot{\xi}_0$ being the velocity at $x = 0$. Within any small element of length δl, measured inwardly from the opening, the variation in velocity is negligible, consequently the kinetic energy is, in this portion of the tube,

$$\delta T = \tfrac{1}{2}\rho S\,\delta l\cdot\dot{\xi}_0^2,$$

if $S = \pi R^2$, the section of the tube. The kinetic energy in the region just outside the opening is, from (198*a*) for *low frequencies*

$$T' = \tfrac{1}{2}Sa_1\dot{\xi}_0^2 = \tfrac{1}{2}S\cdot\frac{8R\rho}{3\pi}\cdot\dot{\xi}_0^2.$$

The total kinetic energy in the region considered, near the opening is therefore

$$\delta T + T' = \tfrac{1}{2}\rho S\dot{\xi}_0^2\left(\delta l + \frac{8R}{3\pi}\right). \tag{200}$$

Thus assuming a uniform velocity in the neighborhood of the opening, the effect of the added mass due to the medium is to prolong the length of the tube by an amount $\alpha = \frac{8R}{3\pi}$. The *effective* length of the pipe is therefore $(l + \alpha)$ and the natural frequencies are given by

$$\cos\frac{\omega(l+\alpha)}{c} = 0, \text{ i.e., } \frac{\omega(l+\alpha)}{c} = \frac{k\pi}{2} \quad (k = 1, 3, 5, \ldots) \tag{201}$$

which is similar to the simple result of eq. (137). It is evident, in the approximate theory given, that the harmonic relations of the simple theory between the overtones of lower frequency will not be disturbed; but as the frequency rises, and the radius of the tube becomes comparable to the wave length (198*a*) is no longer valid. The result is that α becomes smaller

as the higher overtones are reached and the harmonic relation no longer holds.

If the opening is unduly constricted, in comparison with the main conduit, the theory becomes difficult, but the form of the result can be inferred from general principles. There will be increased kinetic energy in the neighborhood of the opening; consequently the added mass will be enhanced, and there will be a greater end correction; there will moreover be a large reflection coefficient for the waves in the tube as they arrive at the constricted end.

The case of an *unflanged* pipe also involves theoretical difficulties in determining the end correction due to inertia. From experiments on flanged and unflanged pipes, Rayleigh estimates (II, § 314) that the effect of the flange is to raise the end correction by about $.22R$; hence, subtracting this value from $\alpha = \frac{8R}{3\pi} = .85R$, the end correction due to inertia for the unflanged opening is about $0.6R$; this is supported by Blaikley's determination ($0.576R$), also quoted by Rayleigh.

In comparing the low-frequency radiation *resistance* of the piston and the hemisphere [equations (196) and (175*a*), following] we have found that the shape of the source was of little importance, as the same result was obtained in both cases, for radiation into the semi-infinite medium. We are consequently justified in assuming that for the *unflanged* end of a tube the radiation resistance is, at low frequencies, substantially that of an isolated point source, as in (175); that is, it is half as great for complete spherical divergence as for divergence into the semi-infinite medium. It is important to recognize that concentration of radiation in one direction is the essence of efficiency in generating sound waves.

Another application of the inertia reaction on the piston, though not giving exact results, is worthy of notice. In the theory of resonators we have stated that the conductivity of a circular opening in a thin wall is $2R$, the mass coefficient being

$$Sa_1 = \frac{\rho S^2}{2R} = \frac{\rho\pi^2 R^3}{2}. \tag{202}$$

The exact derivation of this result is not a subject for simple theory; but the reader may possibly have more confidence in it if we can obtain, on simple grounds, a rough confirmation. Imagine a massless plane piston in the opening, the effect of the piston being to make the *velocity uniform* in all parts of the opening. Now adding together the inertia reactions on *both* sides of the piston, which are equal in all respects, we have twice the coefficient of (198), namely

$$Sa_1 = \rho \tfrac{16}{3} R^3. \tag{202a}$$

The coefficients in (202) and (202*a*) are in the ratio 4.93 : 5.32, or within 8 per cent of one another. The discrepancy is largely due to the fact that in the *actual* case the velocity is *not uniform* over the opening: the more refined theory takes this into account. As in § 24, we again refer the reader to Rayleigh (II, § 306; also *ibid.*, Appendix A) for the exact theory. The result obtained on this basis for the correction for the open end of a flanged tube *is very accurately* $\alpha = .82R$. With this factor determined, the mass factor for the circular aperture is within 5 per cent of twice the added mass at the flanged end of the tube of equal sectional area.

44. *Characteristics of Horns; Conical Horns*

The theory has now reached the point where, if *horns* were not available, it would be our plain duty to invent them, in order to put to practical use the principles of sound radiation which we have established. It must be clear to the reader who has grasped the theory, that the function of the common flaring horn is two-fold. In the narrow portion, which is coupled to the loud-speaking receiver, or other form of sound generator, the source is made to work at maximum efficiency, through the expedient of taking off the radiation in the form of *plane waves*, for under these conditions the radiation resistance of a piston source is a maximum. By flaring out to a large diameter at the open end, the effect is to replace the source (which of itself has very small area) by a large, nearly *flat source of equal rate of*

working, which is better adapted to radiate into the infinite medium. By making the open end of the horn as large as is practicable, without essentially destroying the planeness of the waves within the horn we not only diminish the mass reactance of the medium at the open end, but we also increase the radiation resistance to a value comparable with that for plane waves. Thus in a well-designed horn there is reflection from the open end *only at low frequencies*; transmission to the open medium is accomplished without serious loss, and standing waves or natural vibrations within the horn itself, which give rise to unpleasant resonant effects, are reduced to a minimum.

Considering the horn to be a tube of varying section the basis of the theory was stated by Rayleigh (II, § 265) but it is curious to note that horns have become an accomplished fact largely through the application of empirical methods, rather than as the logical result of the classical theories.

Taking S as the variable section of the tube (i.e., as a function of x, the distance along the axis) the equation of continuity (150) becomes

$$S\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}(S\dot{\xi}) = 0, \tag{203}$$

and since, by (149) and (146) we have

$$\dot{\phi} = c^2 s \qquad \text{and} \qquad \dot{\xi} = -\frac{\partial \phi}{\partial x},$$

the equation of propagation is

$$\ddot{\phi} - \frac{c^2}{S}\frac{\partial}{\partial x}\left(S\frac{\partial \phi}{\partial x}\right) = 0. \tag{204}$$

This may be written in the form

$$\ddot{\phi} = c^2\frac{\partial^2 \phi}{\partial x^2} + c^2\frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}(\log S). \tag{204a}$$

This equation is substantially the basis of A. G. Webster's theory (*Proc. Nat. Acad. Sci.* 5, (1919) p. 275) and has been used by several later writers. It is to Webster that we owe the idea of "Acoustical Impedance" which he had made use of for

some time prior to 1919; the paper just mentioned is important as the starting point for horn theory generally.

Consider first a horn in the shape of a cone, the origin being taken at the vertex. In this case $S = \Omega x^2$ if Ω is the solid angle of the cone. Then

$$\log S = \log \Omega + 2 \log x,$$

and

$$\frac{\partial}{\partial x}(\log S) = \frac{2}{x},$$

so that (204*a*) becomes

$$\ddot{\phi} = c^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{2c^2}{x}\frac{\partial \phi}{\partial x} \quad \text{or} \quad \frac{\partial^2}{\partial t^2}(x\phi) = c^2 \frac{\partial^2}{\partial x^2}(x\phi), \qquad (205)$$

which is identical with (153*a*). On the principles used in the solution of (153*a*), for periodic motions ($\ddot{\phi} = -\omega^2 \phi$) the velocity potential is

$$\phi = \frac{A'}{x} e^{i\omega\left(t - \frac{x}{c}\right)} + \frac{B'}{x} e^{i\omega\left(t + \frac{x}{c}\right)}. \qquad (206)$$

For a diverging wave due to a source of strength A situated at the vertex we should have, analogously to (160)

$$\phi = \frac{A}{\Omega x} e^{i(\omega t - kx)}, \qquad (206a)$$

since the radiation diverges to fill a solid angle Ω instead of 4π. Equation (206*a*) is given by Rayleigh (II, § 280) principally for the purpose of showing how the energy from a source is concentrated by a megaphone.

It is clear that if a conical horn is to be of much use, in raising the efficiency of radiation from the source, the solid angle Ω will necessarily be small. Under these conditions any *finite* conical horn will have little flare, and will therefore owing to reflection at the open end possess resonance characteristics which do not differ essentially from those of tubes of uniform section. The effective length of the tube will be subject to an end correction (dependent on the size of the opening and the

frequency) which is similar to that obtained for the cylindrical tube. Hence the utility of this form of horn is limited.

Important principles relating to the design of horns can, however, be visualized if we consider a very long conical horn, and have the choice of cutting off the horn at a variable distance x_1 from the vertex, a piston source of constant strength then being fitted to the opening at $x = x_1$. This system is equivalent to a conical element of the pulsating sphere system of (§ 37). If we let $S_1 = \Omega x_1^2$, the strength of the source is $A = \dot{\xi}_0 S_1 = \Omega x_1^2 \dot{\xi}_0$; and from (171) the velocity potential at any point $x > x_1$ is, neglecting dissipation due to friction,

$$\phi = \frac{A}{\Omega x} \cdot \frac{(1 - ikx_1)}{1 + k^2 x_1^2} \cdot e^{i\omega t} e^{ik(x_1 - x)}, \tag{207}$$

of which we shall retain only the real portion, that is, the portion due to the motion $A \cos \omega t$ at the source. The excess pressure at a distant point x is the real portion of

$$p = \rho\dot{\phi} = \frac{A\rho\omega}{\Omega x} \frac{(kx_1 + i)}{(k^2 x_1^2 + 1)} e^{i\omega t} e^{ik(x_1 - x)}, \tag{208}$$

that is,

$$p = \frac{A\rho\omega}{\Omega x}\left(\frac{kx_1 \cos[\omega t - k(x - x_1)] - \sin[\omega t - k(x - x_1)]}{(k^2 x_1^2 + 1)}\right),$$

or

$$p = \frac{A\rho\omega}{\Omega x\sqrt{k^2 x_1^2 + 1}} \cos[\omega t - k(x - x_1) + \theta] \tag{208a}$$

$$\left(\theta = \tan^{-1}\frac{1}{kx_1}\right);$$

and the intensity is, since $S = \Omega x^2$,

$$\frac{dW}{dt} = \frac{p^2_{\text{max.}}}{2\rho c} = \frac{A^2 \rho c}{2S\Omega} \frac{k^2}{(1 + k^2 x_1^2)}. \tag{209}$$

For a long conical horn, whose opening S is sufficiently great to "match" the impedance of the infinite medium we are warranted in the following conclusions:

1. For a source of given strength at a fixed position $x = x_1$, the intensity in the neighborhood of the mouth of the horn varies inversely with the *square* of the angle of the cone.

2. At low frequencies (i.e., for $kx_1 < 1$) the intensity rises with the square of the frequency.

3. At high frequencies the intensity approaches asymptotically a value which is *inversely* proportional to the area (Ωx_1^2) of the section of the cone at the source.

4. The larger x_1 can be made, the more nearly will the horn

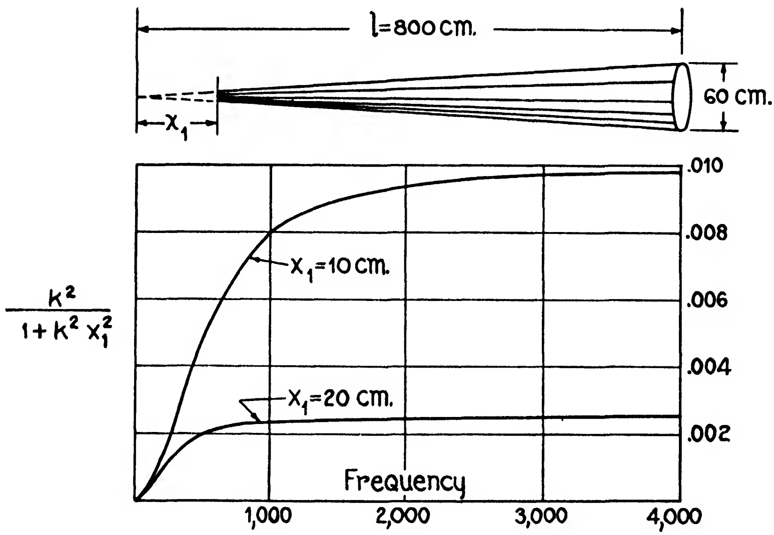

FIG. 16.—RELATIVE INTENSITIES AT THE MOUTH OF CONICAL HORN (NEGLECTING REFLECTION).

afford distortionless transmission for all frequencies. The interpretation of Fig. 16, which shows the frequency variation of intensity for two values of x_1 in the case of long horn is left to the reader.

Two major factors enter into the design of the conical horn. The length should be as great as is practicable, for thereby we have at once the possibility a large area S at the mouth and a small angle Ω; the quantity $S\Omega$ is kept within reasonable bounds. If S is large, resonance due to reflection is minimized,

for reasons which we have previously given. In addition, if the length is great the more serious natural vibrations will occur at lower frequency where their effects are least harmful. Thus all considerations of efficiency require a long horn, if it is in the form of a cone.

The second important factor is the value of x_1; this will be determined by a compromise depending on how much we wish to sacrifice efficiency at high frequencies for the sake of extending transmission to as low a frequency as possible and so obtaining a more nearly uniform frequency characteristic.

In practice a very important question is that of properly adapting the loud-speaking receiver element to the horn. This is a specialized study, quite outside the limits of this text: but one matter of importance may be mentioned. With any horn that is designed to radiate properly over a wide range of frequency from a source of uniform strength at all frequencies, it can be safely assumed that it will behave well when driven by such vibrating systems as are likely to be fitted to it. The radiation damping provided by the horn will usually be sufficiently great to smooth out any moderate amount of resonance in the driving system. Indeed resonance to a certain degree in the driving system may be used to advantage to improve the transmission in the low frequency region. This principle has been applied in the design of the piston loud speaker, which has no horn; but it is equally applicable to sound generators [1] which are used in connection with horns. The poor transmission which is usually afforded at low frequencies is not necessarily inherent in vibrators or horns, though it may be aggravated if these are not properly designed; it is the fundamental difficulty (which we first encountered in the problem of the pulsating sphere) of driving the medium at low frequencies if divergence of the radiation is permitted at the source.

The results obtained in this article are equivalent to those given by E. W. Kellogg (*Gen. Elec. Rev.*, XXVII, Aug., 1924, p. 556). The theory of the conical horn of finite length is given

[1] For references to loud-speaking apparatus of various types, see Appendix B.

by Webster, in the paper cited; it has been somewhat extended by G. W. Stewart (*Phys. Rev.*, XVI, Oct., 1920, p. 313).

45. *Flaring Horns of Exponentially Varying Section*

The flaring horn, of exponentially increasing section, is an instrument of such great utility that it merits special attention.

Taking the origin at the small end, let $S = S_1 e^{mx}$ and we have

$$\frac{\partial}{\partial x}(\log S) = m,$$

so that, for periodic motions ($\ddot{\phi} = -\omega^2\phi$) equation (204*a*) becomes

$$\left(\frac{\partial^2}{\partial x^2} + m\frac{\partial}{\partial x} + k^2\right)\phi = 0; \qquad \left(k = \frac{\omega}{c} = \frac{2\pi}{\lambda}\right). \tag{210}$$

The solution of this equation, by the method of § 3 (p. 9), is

$$\phi = A_1 e^{\mu_1 x} + B_1 e^{\mu_2 x}, \tag{211}$$

in which

$$\left.\begin{aligned} \mu_1 &= -\frac{m}{2} + \frac{i}{2}\sqrt{4k^2 - m^2} = -\alpha + i\beta, \\ \mu_2 &= -\frac{m}{2} - \frac{i}{2}\sqrt{4k^2 - m^2} = -\alpha - i\beta, \end{aligned}\right\} \tag{212}$$

analogously to equations (5). The velocity potential is therefore the real part of

$$\phi = e^{-\alpha x}[A_1 e^{-i\beta x} + B_1 e^{+i\beta x}]e^{i\omega t}, \tag{213}$$

that is, for wave motion in both directions,

$$\phi = \epsilon^{-\alpha x}[A_1 \cos(\omega t - \beta x) + B_1 \cos(\omega t + \beta x)]. \tag{213a}$$

For a long horn, neglecting reflection at the open end (which is presumably of large area) we should have for a divergent wave,

$$\phi = A_1 e^{-\alpha x}\cos(\omega t - \beta x). \tag{213b}$$

This equation is similar to (124) § 32 in its kinematical properties; the only difference being that the attenuation in (213*b*) is

due not to friction (which we have neglected) but to the lateral release of pressure as the wave diverges to fill the horn. In both (124) and (213*b*) the result of attenuation is not only to diminish the amplitude, but to change slightly the velocity of propagation. In discussing (124) this latter effect was neglected as it was not essential in the treatment of the tube problem; but it cannot legitimately be neglected here. If there were no attenuation, the velocity of propagation would be $\frac{\omega}{k} = c$; but under the conditions which exist in the horn we must have, for *sustained* wave motion.

$$c' = \frac{\omega}{\beta} = \frac{\omega}{k} \cdot \frac{1}{\sqrt{1 - \frac{m^2}{4k^2}}} = \frac{c}{\sqrt{1 - \frac{m^2}{4k^2}}}. \qquad (214)$$

The velocity potential is thus slightly *advanced* in phase as compared with that of a plane wave, assuming no dissipation. It is apparent, on differentiating ϕ with respect to x, and with respect to t, that the velocity and the excess pressure are no no longer in phase, as they are in the case of plane waves (§ 31, p. 91). Hence we must proceed with some care in using these quantities to find the intensity of the radiation.

The immediate problem is to compare the frequency variation of the intensity at the mouth of a long "exponential" horn with that for a conical horn for which the ratio of initial to final section is the same. It is convenient in this problem to take the velocity as the basis of phase, as we shall make the velocity at the origin a prescribed quantity. As before, we have

$$\dot{\xi}_{x=0} = \frac{A}{S_1} \cos \omega t,$$

taking A as the strength of the source. It can then be shown that the velocity potential due to this source is

$$\phi = \frac{Ae^{-\alpha x}}{S_1 k^2}[\alpha \cos (\omega t - \beta x) + \beta \sin (\omega t - \beta x)], \qquad (215)$$

and the velocity at any point x, is

$$\dot{\xi} = -\frac{\partial \phi}{\partial x} = \frac{Ae^{-\alpha x}}{S_1} \cos(\omega t - \beta x), \tag{216}$$

since from (212), $\alpha^2 + \beta^2 = k^2$. This obviously satisfies the condition imposed at the origin. The excess pressure is

$$p = \rho\dot{\phi} = \frac{\rho\omega Ae^{-\alpha x}}{S_1 k^2}[-\alpha \sin(\omega t - \beta x) + \beta \cos(\omega t - \beta x)]. \tag{217}$$

We shall not assume (though it would be legitimate, with a simple reservation, as we shall find) that the intensity is given by the formula (164*a*) which we have repeatedly used for plane and spherical waves. Returning to first principles, we have, for the intensity

$$\frac{dW}{dt} = \dot{\xi}\cdot p$$

$$= \frac{\rho\omega A^2 e^{-2\alpha x}}{k^2 S_1^2}[\beta \cos^2(\omega t - \beta x) - \alpha \sin(\omega t - \beta x) \cos(\omega t - \beta x)],$$

or, taking mean values

$$\left|\frac{dW}{dt}\right| = \overline{\dot{\xi}\cdot p} = \frac{\rho c A^2 e^{-2\alpha x}\cdot\beta}{2kS_1^2}, \tag{218}$$

since $\frac{\omega}{k} = c$.

We have already noted the relation $\alpha^2 + \beta^2 = k^2$; α may therefore be placed equal to $k \sin \theta$, and β to $k \cos \theta$. Except at very low frequencies α is a small quantity; it may be taken as a measure of the phase difference between the velocity and the excess pressure. The quantity $\cos \theta = \frac{\beta}{k}$ may be looked upon as a "power factor."

We may now remark that equation (218) would have resulted if we had written, instead of (217), the equivalent form

$$p = \frac{\rho\omega Ae^{-\alpha x}}{S_1 k} \cos(\omega t - \beta x + \theta), \tag{217a}$$

and used for the intensity the relation similar to (164*a*)

$$\frac{dW}{dt} = \frac{(p_{\max.})^2}{2\rho c'},$$

in which c' is taken, as in (214), equal to $\frac{c}{\cos\theta}$.

The form of equation (218) must be changed slightly to compare it with the corresponding equation (209) which was obtained for the conical horn. We have

$$e^{-2\alpha x} = e^{-mx} = \frac{S_1}{S} \quad \text{and} \quad \beta = k\sqrt{1 - \frac{m^2}{4k^2}}, \tag{219}$$

so that (218) becomes, on substitution

$$\frac{dW}{dt} = \frac{\rho c A^2}{2SS_1}\sqrt{1 - \frac{m^2c^2}{4\omega^2}}. \tag{218a}$$

The importance of the quantity β in the theory of the exponential horn is clear. For a certain frequency (i.e., when $2\omega = mc$) $\beta = 0$; for frequencies *below this critical value*, the horn transmits nothing, or in other words behaves as a filter. As the frequency rises above the critical frequency, β rises rapidly, $\cos\theta$ asymptotically approaching the value unity.

If we note that $\Omega = \frac{S_1}{x_1^2}$, equation (209) for the cone may be written

$$\frac{dW}{dt} = \frac{\rho c A^2}{2SS_1}\frac{k^2x_1^2}{(1 + k^2x_1^2)}, \tag{209a}$$

which permits the immediate comparison of the two types of horns, as A, S, and S_1 are the same in both cases. If it is not evident at once, that (except below the critical frequency) the exponential horn is much superior, an example will make the matter clear. In Fig. 17 are shown the two frequency characteristics, according to (218*a*) and (209*a*); the length of the horns being taken as 192 cm. and the ratio $\frac{S}{S_1}$ being $(25)^2 = 625$.

From this ratio, the rate of taper for the exponential horn is determined to be $m = .033$; the only other quantity required in the calculations is $x_1 = 8$ cm., the length measured from the vertex of the cone to the point where the section is S_1. The figure should be self-explanatory.

The conclusion we have reached as to the superiority of the exponential horn amply justifies its widespread application in loud-speaking apparatus. The fundamental equations of the

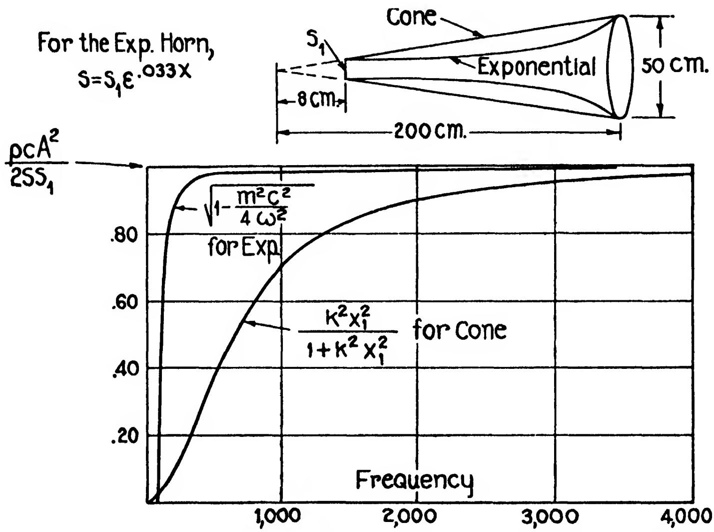

FIG. 17.—COMPARISON OF CONICAL AND EXPONENTIAL HORNS HAVING SAME INITIAL AND FINAL OPENINGS.

exponential horn were given by A. G. Webster (*loc. cit.*). More recently the theory has been considered by Hanna and Slepian (*Trans. A.I.E.E.*, 43, 1924, p. 393), and by H. C. Harrison (British Patent No. 213,528, 1925). In the paper by Goldsmith and Minton on Horns (*Proc. Inst. Rad. Eng.*, Aug., 1924, p. 423) these writers are apparently *in error* (p. 454) in finding that a given conical horn has better transmission characteristics for certain frequencies above the critical frequency than the corresponding exponential horn. We cannot agree with their conclusion.

Horns have been extensively studied by P. B. Flanders, from whose most important memorandum the following quotation is taken, for the purpose of summarizing this article. The statement bears on the application of a horn to the short tube leading from the loud-speaking receiver diaphragm.

"Neglecting reflection effects, the addition of a horn does not effect an increase in the *ultimate* impedance at the end of a receiver opening. It does, however, cause that impedance to reach its ultimate value at a lower frequency; and the lower that frequency, the better, of course is the horn. The impedance "looking out" of a seven-tenths inch hole in an infinite wall reaches 80 per cent of its ultimate value at 9300 c.p.s.; a certain conical horn causes this 80 per cent value to be reached at 4200 c.p.s.; while the corresponding exponential horn causes this value to be reached at the relatively low frequency of 250 c.p.s. In a way these figures show why the exponential horn is so much superior to the conical type."

46. *The Finite Exponential Horn*

The theory of horns, to be of practical use, must be applied to horns of finite length; this problem we shall now consider, taking for example the exponential horn because of its superiority. The method is straightforward, being based on that already used in (§ 32) for the tube problem; but the calculations are laborious, though some simplification is possible if we make use of the impedance-methods which are well-known in connection with the study of electrical networks.[1] We shall first find the principal impedances which are characteristic of finite and infinite horns, and then substitute for the given horn an equivalent network based on these constants. It will then be possible to find how the given horn behaves when placed between two given impedances, with a given driving force ap-

[1] The method of applying the theory of electrical networks to the horn, as used in this text, is due to P. B. Flanders and D. A. Quarles of the Bell Telephone Laboratories. The advantages of the method will be evident, if a comparison is made between the present text, and the treatment of the horn by A. G. Webster, cited in § 44.

plied at one end. Finally it will be necessary to make certain approximations in dealing with the impedance of the medium at the large (open) end; but this is not a matter of great difficulty in view of the theory given in the earlier part of this chapter.

From the general solutions already obtained [equations (216) and (217*a*)] we may take for the velocity and the excess pressure

$$\dot{\xi} = e^{-\alpha x}[Ae^{-i(\beta x+\theta)} - Be^{i(\beta x+\theta)}]e^{i\omega t}, \tag{220}$$

and

$$p = \rho c e^{-\alpha x}[Ae^{-i\beta x} + Be^{i\beta x}]e^{i\omega t}, \tag{221}$$

in which, as before, $\theta = \tan^{-1}\dfrac{\alpha}{\beta}$, the angle of lag of velocity with respect to pressure, for a wave travelling in the positive direction—that is, the direction of increasing section of the horn. Equations (220) and (221) thus provide for all possible waves in both directions, θ reversing its sign for propagation in the negative direction.

We have at once for the impedance of an *infinite horn*, in the *positive* direction

$$Z = \frac{Sp}{\dot{\xi}} = \rho c S e^{i\theta} = \frac{\rho c S}{k}(\beta + i\alpha), \tag{222}$$

and similarly, in the *negative* direction (since then c is negative)

$$\bar{Z} = \rho c S e^{-i\theta} = \frac{\rho c S}{k}(\beta - i\alpha). \tag{222a}$$

Now let the horn be terminated at $x = 0$ and $x = l$ with piston impedances Z_0 and Z_l respectively. The boundary conditions are thus:

$$\text{at} \quad x = 0, \qquad Z_0\dot{\xi}_1 + p_1 S_1 = \Psi_0 e^{i\omega t}, \tag{223a}$$

$$\text{at} \quad x = l, \qquad Z_l\dot{\xi}_2 - p_2 S_2 = 0; \tag{223b}$$

in which $\Psi_0 e^{i\omega t}$ is the applied force, and the subscripts 1, 2, refer to local values of velocity, pressure and section. Applying

the condition (223*b*) to equations (220) and (221), we find the relation

$$B = Ae^{-i2\beta l}\frac{[Z_l e^{-i\theta} - \rho c S_2]}{[Z_l e^{i\theta} + \rho c S_2]}. \qquad (224)$$

Eliminating B between (224) and (223*a*), we have

$$A = \frac{[Z_l e^{i\theta} + \rho c S_2]\Psi_0}{2De^{-i\beta l}}, \qquad (225)$$

in which

$$2D = (Z_0 Z_l + \rho^2 c^2 S_1 S_2)(e^{i\beta l} - e^{-i\beta l})$$

$$+ \rho c[Z_0 S_2(e^{i(\beta l-\theta)} + e^{-i(\beta l-\theta)}) + Z_l S_1(e^{i(\beta l+\theta)} + e^{-i(\beta l+\theta)})],$$

that is

$$D = (Z_0 Z_l + \rho^2 c^2 S_1 S_2) i \sin \beta l$$

$$+ \rho c[Z_l S_1 \cos(\beta l + \theta) + Z_0 S_2 \cos(\beta l - \theta)]. \qquad (226)$$

We also have

$$B = \frac{e^{-i\beta l}[Z_l e^{-i\theta} - \rho c S_2]\Psi_0}{2D}. \qquad (227)$$

Inserting in (220) and (221) the values for A and B, we have for the velocity and pressure distributions in the horn,

$$\dot{\xi} = \frac{e^{-\alpha x}}{D}(Z_l \cdot i \sin \beta(l-x) + \rho c S_2 \cos[\beta(l-x) - \theta])\Psi_0 e^{i\omega t}, \quad (228)$$

and

$$p = \frac{\rho c e^{-\alpha x}}{D}(\rho c S_2 \cdot i \sin \beta(l-x) + Z_l \cos[\beta(l-x) + \theta])\Psi_0 e^{i\omega t}. \quad (229)$$

We can now construct the equivalent T-network for the horn, considering it (in the positive direction) to have the pure inertia coefficients L_1, M and L_2 just as an ordinary transformer. These relations will be clear if we refer to Fig. 18. For the transformer $i\omega L_1$ is the driving-point impedance at (1), with the secondary circuit open; this corresponds to the driving

point impedance of the horn at (1) when (2) is rigidly closed so that $\dot{\xi}_2 = 0$. The quantity $i\omega L_2$ is determined similarly. The mutual impedance $M_{12} \equiv i\omega M$ is the ratio, for rigid closure of the horn at S_2, of the force $(p_2 S_2)$ to the velocity $\dot{\xi}_1$. M is also given by the simple formula

$$M = \sqrt{L_2\left(L_1 - \frac{L_2 Z_1'}{i\omega}\right)}, \qquad (230)$$

in which Z_1' is the impedance at (1) with (2) electrically short circuited, or in the acoustic case, with the impedance $Z_l = 0$.

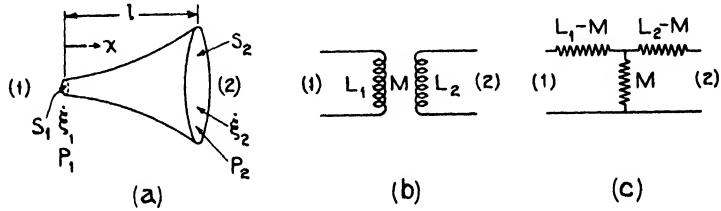

FIG. 18.—FINITE HORN, AND NETWORK EQUIVALENT.

To determine L_1, we put $Z_l = \infty$ in (228) which becomes, for $x = 0$,

$$\dot{\xi}_1 = \frac{i \sin \beta l \cdot \Psi_0 e^{i\omega t}}{Z_0 \cdot i \sin \beta l + \rho c S_1 \cos (\beta l + \theta)}. \qquad (231)$$

The driving-point impedance is therefore

$$\frac{\Psi_0}{(\dot{\xi}_1)_{\max.}} = Z_0 + \frac{\rho c S_1 \cos (\beta l + \theta)}{i \sin \beta l}.$$

Subtracting the piston impedance Z_0, we have for the impedance of the horn itself,

$$Z_1 = i\omega L_1 = \frac{i\rho c S_1}{k}(\alpha - \beta \cot \beta l), \qquad (232)$$

since $\cos \theta = \frac{\beta}{k}$ and $\sin \theta = \frac{\alpha}{k}$. To find L_2, we can avoid the labor of redetermining the constants A and B, for force applied at point (2), which would involve a new formula in which $(-x)$ was taken as the positive direction; we merely note that in so

changing our point of view, θ becomes $-\theta$ (pressure lagging velocity) hence $\alpha(=k\sin\theta)$ becomes $(-\alpha)$ while β is unchanged. We then have

$$Z_2 = i\omega L_2 = \frac{-i\rho c S_2}{k}(\alpha + \beta\cot\beta l). \tag{233}$$

We proceed to determine M. Repeating (231), for $Z_0 = 0$,

$$\dot{\xi}_1 = \frac{i\sin\beta l\cdot\Psi_0 e^{i\omega t}}{\rho c S_1\cos(\beta l + \theta)}. \tag{231a}$$

The pressure at (2) is, for $Z_0 = 0$ and $Z_l = \infty$, from (22)

$$p_2 = \frac{e^{-\alpha l}\cos\theta\cdot\Psi_0 e^{i\omega t}}{S_1\cos(\beta l + \theta)}, \tag{234}$$

and the force is

$$S_2 p_2 = \frac{\beta}{k}\frac{\sqrt{S_1 S_2}\Psi_0 e^{i\omega t}}{S_1\cos(\beta l + \theta)}, \quad \text{since } e^{-\alpha l} = \sqrt{\frac{S_1}{S_2}}. \tag{235}$$

The mutual impedance is, by definition,

$$M_{12} = i\omega M = \frac{S_2 p_2}{\dot{\xi}_1} = -i\rho c\frac{\beta}{k}\frac{\sqrt{S_1 S_2}}{\sin\beta l}. \tag{236}$$

The reader may verify this equation independently by determining Z_1' and applying equation (230). This will incidentally verify the assumed equivalent T-network shown in Fig. 18.

In the discussion above, both α and β have been considered to be real quantities; that is, the frequency has been taken to be within the transmission range. We shall not deal with phenomena in the horn for frequencies below the critical frequency, for which β becomes imaginary, because in practice the critical frequency is always very low. The point is mentioned, however, because in any complete theory of the exponential horn, the whole range of frequencies must be considered.

To get a correct idea of the essential characteristics of the finite horn, we must choose terminal impedances at the ends of the network with some regard for actual operating conditions, and we must assume a method of driving at the point (1) which

does not unduly distort the normal characteristics of the horn. To avoid the complication of considering the characteristics of the source of sound applied to the horn, we shall simply assume that the fluid in the small end of the horn is driven with constant periodic force, by means (say) of a piston whose impedance is negligible. At the large end of the horn, we shall take the impedance offered by the medium to be

$$Z_l = \rho c S_2(X + iY), \tag{237}$$

in which X and Y are functions of frequency of the nature of those discussed in § 42 [eq. (193*a*)], when dealing with the reactions on a piston. The particular form of $X + iY$ will be determined later.

Making use of the equivalent network of the horn we have, for the impedance at the driving point,

$$Z = \frac{Z_1Z_2 - M_{12}^2 + Z_1Z_l}{Z_2 + Z_l}, \tag{238}$$

the velocity at the driving point being

$$\dot{\xi}_1 = \frac{\Psi}{Z}, \tag{238a}$$

if Ψ is the applied force, ($\Psi_0 e^{i\omega t}$) independent of the frequency. Using the relations (232), (233), and (236) it can be shown that the expression, in (238),

$$Z_1Z_2 - M_{12}^2 = \rho^2c^2S_1S_2. \tag{239}$$

If the horn were driven at constant periodic velocity, we should use, to compute the velocity at the large end, the relation

$$\dot{\xi}_2 = \frac{M_{12}\dot{\xi}_1}{Z_2 + Z_l}, \tag{240}$$

but we shall see that this is not the best point of view from

which to regard the horn. Substituting the values for M_{12}, Z_2 and Z_l [equations (233,) (236,) (237)] we have

$$\dot{\xi}_2 = \frac{-\,i\cos\theta\cdot\sqrt{S_1S_2}\cdot\dot{\xi}_1}{S_2(X\sin\beta l + i[Y\sin\beta l - (\sin\theta\sin\beta l + \cos\theta\cos\beta l)])}. \tag{241}$$

When $\beta l = m\pi$, m being integral, this reduces to

$$\dot{\xi}_2 = \pm\dot{\xi}_1\sqrt{\frac{S_1}{S_2}} \quad \text{(condition } a\text{)}, \tag{241a}$$

and similarly, if $\beta l = (2m+1)\frac{\pi}{2}$, (condition b),

$$\dot{\xi}_2 = \frac{\mp\, i\cos\theta}{(X+iY) - \sin\theta}\,\dot{\xi}_1\sqrt{\frac{S_1}{S_2}}. \tag{241b}$$

It is now clear that, for constant driving velocity $\dot{\xi}_1$, no account is taken [at least in condition (a)] of the output impedance $Z_l = \rho c S_2\,(X+iY)$ of the horn; in other words, in studying $\dot{\xi}_2$ as dependent on $\dot{\xi}_1$, no weight is given to the difficulty of keeping $\dot{\xi}_1$ constant, due to the varying of the driving-point impedance with frequency. In passing we note that condition (a) corresponds to the case of an organ pipe open at both ends [eq. (141), § 33] and is a condition for maximum velocity at both ends of the horn, due to resonance; for the case of a simple tube it was shown in § 32 that this was also the condition for maximum efficiency in power transmission, the driving-point impedance being then the sum of that of the driving piston and of the piston at the other end of the tube. This is of course true here for the horn, as is evident also from equation (244a) below.

If on the other hand we wish to drive the horn with constant periodic force Ψ, we use the relation [1]

$$\dot{\xi}_2 = \frac{M_{12}\Psi}{(Z_1Z_2 - M_{12}^2) + Z_1Z_l}. \tag{242}$$

[1] In (242) and similar equations following, the reciprocal of the coefficient of Ψ is of course the "transfer impedance" of the finite horn. (242) follows from (238a) and (240).

Referring to (239), for the expression in parentheses, and noting that

$$Z_1 Z_l = \frac{\rho^2 c^2 S_1 S_2}{k}[Y(\beta \cot \beta l - \alpha) - iX(\beta \cot \beta l - \alpha)], \quad (243)$$

we have

$$\dot{\xi}_2 = \frac{-i\rho c \cos\theta \sqrt{S_1 S_2}\cdot\Psi}{\rho^2 c^2 S_1 S_2[\sin\beta l + (Y - iX)(\cos\theta\cos\beta l - \sin\theta\sin\beta l)]}.$$

If $\beta l = m\pi$ (m an integer, as before)

$$\dot{\xi}_2 = \frac{\pm\Psi}{\rho c\sqrt{S_1 S_2}(X + iY)}, \quad (244a)$$

and if $\beta l = (2m + 1)\frac{\pi}{2}$,

$$\dot{\xi}_2 = \frac{\mp\Psi\cos\theta}{\rho c\sqrt{S_1 S_2}[(X + iY)\sin\theta - i]}. \quad (244b)$$

We may now deal with the terminal impedance (i.e., the impedance of the medium at the open end) which is of the form

$$\rho c S_2(X + iY). \quad (237)$$

If this were the impedance offered to a piston *in an infinite wall*, we should have (eq. 193*a*)

$$X = \left[1 - \frac{J_1(2kR)}{kR}\right], \qquad Y = \frac{K_1(2kR)}{2k^2R^2}, \quad (245)$$

the added inertia due to the medium at zero frequency being

$$\left.\frac{i\rho c S_2 Y}{i\omega}\right|_{\omega=0} = \rho S_2\cdot\frac{8}{3\pi}\sqrt{\frac{S_2}{\pi}}, \quad (245a)$$

from (198). Now, inasmuch as we have no solution of the problem of the reactions on a piston oscillating in the end of a *non-flanged tube*, it is necessary to make some such assumption as that of eq. (245), in order to obtain a manageable theoretical solution of the problem of the finite horn. It seems reasonable that the approximation involved will be sufficiently close for the purpose, particularly since we have tacitly made other

simplifying assumptions to obtain the theory in the simple form given. (For example, no allowance has been made for dissipation; and no allowance has been made for the fact that the wave front may have such a curvature that the conditions we have assumed as to its expanding area as it proceeds may not be quite fulfilled.) However, if it is tolerable to solve the problem as if the end of the horn were fitted with an infinite plane flange, then it seems reasonable to go one step further, and make the flange a very wide-angled cone. If this is done, we can then consider the wave front, as it emerges from the open end of the horn, to be equivalent to that produced by pulsating spherical surface. The question then arises as to what the area and curvature of the spherical surface should be to match properly; if this can be answered readily, we shall have available a much simpler function for $X + iY$ than that given in (245).

We have already noted [see eq. (175*a*) and remarks following] that a piston and a pulsating hemisphere of equal area, both flanged by an infinite plane, have the same radiation resistance at high and at low frequencies, though for intermediate values, the piston approaches the final maximum more rapidly. We take therefore a spherical surface, whose curvature is to be determined, but whose area is S_2, to be the approximate equivalent of the piston in an infinite wall. To determine the curvature (hence the solid angle of the conical flange) we adjust the solid angle of the cone so that the maximum added inertia (that at zero frequency) is the same in the two cases. We have found [eq. (174)] that the added mass, for a spherical surface of radius r_0 is ρr_0 per unit area at zero frequency. If we let $S_2 = \Omega r_0^2$, we have $r_0 = \sqrt{\frac{S_2}{\Omega}}$, hence equating the mass $\rho r_0 S_2$ for the spherical surface, to that for the piston of equal area (245*a*), we have

$$\rho S_2 \cdot \frac{8}{3\pi}\sqrt{\frac{S_2}{\pi}} = \rho S_2 \sqrt{\frac{S_2}{\Omega}}, \tag{246}$$

and a simple calculation gives $\Omega = 1.39\pi$, for the solid angle. If R is the radius of the piston, then $\pi R^2 = 1.39\pi r_0^2$ and

$r_0 = .85R$. We then have, for the new functions X' and Y'. from (174)

$$X' = \frac{k^2 r_0^2}{k^2 r_0^2 + 1}, \qquad Y' = \frac{k r_0}{k^2 r_0^2 + 1}, \tag{247}$$

since $\omega = kc$, and $k = \frac{2\pi}{\lambda}$. Finally, since the impedance $Z_l = \rho c S_2(X' + iY')$ we may write

$$Z_l = \rho c S_2 \cdot \cos\phi \cdot e^{i\phi}, \qquad \phi = \tan^{-1}\frac{1}{kr_0}. \tag{248}$$

The method suggested for dealing with the impedance at the end of a horn may have possible application to other acoustic problems, hence it is interesting to compare the behavior of the functions X and Y of (245) on the "piston" basis, with those of (247) based on the "equivalent" spherical surface. This is done in Fig. 19, both sets of functions being plotted in terms of the argument $kR = 1.18kr_0$; it will be observed that the greatest discrepancy between X and X' is about 20 per cent (for $kR = 2.5$), while that between Y and Y' is about 40 per cent, for $kR = 1.3$. The theoretical discrepancy is doubtless greater than that which would be observed in any practical case, because the wave front emerging from the large end of the horn can hardly be strictly plane; it is much more likely to be convex. And it may be noted that the smaller value for Y' as compared with Y at low frequencies happens to be more nearly in agreement with the *smaller end correction* for an unflanged tube (as compared with a flanged tube) which was noted by Rayleigh, and mentioned in § 43, p. 151 above.

Placing the expression for Z_l (248) in (244*a*) we have, for the values of ξ_2 corresponding to the condition $\beta l = m\pi$ (dropping the exponential factor)

$$|\xi_2|_a = \frac{\pm\Psi_0}{\rho c\sqrt{S_1 S_2}} \cdot \frac{1}{\cos\phi} \cdot e^{-i\phi}, \quad \left(\frac{1}{\cos\phi} = \frac{\sqrt{k^2 r_0^2 + 1}}{k r_0}\right). \tag{249}$$

For the other condition, rationalizing (244*b*) we have, since $X'^2 + Y'^2 = X'$,

$$\left.\begin{aligned} |\dot{\xi}_2|_b &= \frac{\mp \Psi_0}{\rho c \sqrt{S_1 S_2}} \cdot \frac{\cos\theta \cdot e^{-i\phi'}}{\sqrt{(X'\sin\theta - 2Y')\sin\theta + 1}}, \\ \tan\phi' &= \frac{Y'\sin\theta - 1}{X'\sin\theta}, \end{aligned}\right\} \qquad (250)$$

when $\beta l = (2m + 1)\frac{\pi}{2}$. We can now state the general behavior of the horn, within its transmission range, remembering that for the lowest transmitted frequency $\sin\theta = \frac{\alpha}{k} = 1$, $\cos\theta = 0$, while as higher frequencies are reached $\sin\theta \doteq 0$ and $\cos\theta \doteq 1$. We may also refer again to the upper curve in Fig. 17 which is a plot of $\cos\theta$ for a certain exponential horn, this having been required to show the frequency variation of intensity at the mouth, assuming no reflection there.

We note first that (§ 45, p. 160)

$$\sin\theta = \frac{\alpha}{k} = \frac{m}{2k}; \qquad \cos\theta = \frac{\beta}{k} = \sqrt{1 - \frac{m^2}{4k^2}}, \qquad (251)$$

for the phase angle θ characteristic of the horn. We then have for the frequencies for which (250) is valid (that is for a condition of non-resonance in the horn),

$$X'\sin^2\theta = \frac{m^2 r_0^2}{4(k^2 r_0^2 + 1)}, \qquad Y'\sin\theta = \frac{m r_0}{2(k^2 r_0^2 + 1)}. \qquad (252)$$

It is evident that *for these frequencies* (since the quantities of (252) are small as compared with 1) the velocity $|\dot{\xi}_2|_b$ at the mouth of the horn is approximately the same as if we were dealing with an infinite horn: that is, proportional to $\cos\theta$. At the higher frequencies, as $\cos\theta$ approaches unity, ϕ' approaches zero.

For the frequencies for which $\beta l = m\pi$, which are intercalated between the successive values of frequency dealt with above, the finite horn is in resonance, the values of $|\dot{\xi}_2|_a$ being proportional to $\frac{1}{\cos\phi}$; therefore (cf. 249) these points are a

series of maxima in the response of the horn, which *decrease in height with rising* frequency and asymptotically approach the level

$$|\dot{\xi}_2|_b = |\dot{\xi}_2|_a = \frac{\pm \Psi_0}{\rho c\sqrt{S_1 S_2}}, \tag{253}$$

which is the constant response characteristic of the horn at high frequencies. When this range is reached, resonance is eliminated since the impedance of the horn matches that of the medium; the horn behaves as a simple transformer working at maximum efficiency, and sends forth a nearly parallel beam of sound radiation.

There are many interesting problems in connection with horns, such as, for example, the directive effects in the radiation after it emerges from the mouth; the effects of dissipation; the effects of phase change, etc. Where these can be handled theoretically, they are left to the reader, on the basis of the principles given in this chapter; they are too specialized to consider here. It appears that some of the effects obtained with horns require considerable experimental study in order to elucidate them; unfortunately no discussion of these can be given, for lack of suitable data.

47. *The Effect of Sound Waves on a Simple Vibrating System*

When we dealt with the elementary theory of the resonator, considered as a vibrating system (§ 24), it was necessary to give some account of the behavior of the system when acted on by sound waves. Now, our discussion of typical radiation and transmission problems would be incomplete if we did not consider more fully the phenomena when a resonator or other simple system is immersed in a field of sound waves, because this problem is likely to arise whenever sound waves are to be detected or measured. At the risk of repetition, we shall reopen the question of selective absorption and amplification (this time on an energy basis); in addition we shall give some attention to the distortions of the original field, due to the presence of the resonator; and finally we shall note the effect

of a resonator on the behavior of a nearby source of sound. All these questions are difficult to answer rigorously, particularly the second, because in addition to the purely resonant action of the system, the mounting in which it is contained acts as a rigid obstacle to further distort the field.

Consider again the mechanism of absorption, which is inevitably bound up with re-emission as in all general radiation problems. Suppose (as in § 24) that the resonator is *tuned* to the frequency of the driving waves, which arise from a source of strength A_1 situated at a distance x from the mouth of the resonator. The only impedance offered by the moving mass of air in the resonator opening is its radiation resistance, which is

$$Sb_1 = \frac{\rho\omega^2 S^2}{4\pi c}, \tag{175}$$

if the dimensions of the resonator are small as compared with the wave length. If we assume no resonator present, the velocity potential at the point x is

$$\phi_0 = \frac{A_1}{4\pi x} e^{i(\omega t - kx)} \tag{254}$$

and the excess pressure is

$$p = \rho\dot{\phi}_0 = i\omega\rho\phi_0. \tag{255}$$

The intensity is given by

$$\left|\frac{dW}{dt}\right|_{\text{av.}} = \frac{|p^2|_{\text{max.}}}{2\rho c} = \frac{\omega^2\rho|\phi_0{}^2|_{\text{max.}}}{2c}. \tag{256}$$

Now it is evident that when faced by the impedance (175), (which is much less than the impedance $\rho c S$ of an equivalent area of the free medium) the excess pressure in the oncoming wave will produce a much greater velocity at the point x than if the resonator were absent. The energy must come from the adjoining region of the medium, and the question is, how large is the area from which this excess of energy is abstracted? A rough answer is obtained if we compute the velocity from the

pressure according to (255) above, and the impedance (175). We have then (using constants per unit area)

$$\dot{\xi} = \frac{-i\omega\rho\phi_0}{\rho\omega^2 S}\cdot 4\pi c = \frac{-i\,4\pi\phi_0}{kS}, \tag{257}$$

the resonator now behaving as a new source of strength

$$A_2 = S|\dot{\xi}|_{\text{max.}} = -\frac{4\pi}{k}|\phi_0|_{\text{max.}} = -\frac{iA_1}{kx}. \tag{258}$$

The maximum rate of working is $A_2|p|_{\text{max.}}$ that is,

$$\left.\begin{aligned} |A_2 p|_{\text{max.}} &= \frac{4\pi}{k}\cdot\omega\rho|\phi_0{}^2|_{\text{max.}} = \frac{\lambda^2}{\pi}\cdot\frac{\omega^2\rho}{c}|\phi_0{}^2|_{\text{max.}}, \\ &\qquad\left(k = \frac{\omega}{c} = \frac{2\pi}{\lambda}\right); \end{aligned}\right\} \tag{259}$$

or, on the average

$$|A_2 p|_{\text{av.}} = \frac{\lambda^2}{\pi}\frac{\omega^2\rho}{2c}|\phi_0{}^2|_{\text{max.}} = \frac{\lambda^2}{\pi}\left|\frac{dW}{dt}\right|_{\text{av.}} \tag{259a}$$

by comparison with (256). From this it appears that the area of wave-front from which energy is abstracted (and reradiated), by the resonator is of the order of magnitude $\frac{\lambda^2}{\pi}$. That is to say, the energy density in the area S of the resonator mouth is greater than that in the original field, in the ratio $\frac{\lambda^2}{\pi S}$, this energy-amplification coefficient being exactly that previously obtained in § 24 for the velocity amplification coefficient. This is as it should be, because the driving force (Sp) is not in any way amplified by the resonator; the rate of working ($p\dot{\xi}$) is merely increased linearly with the increase in velocity. The area $\frac{\lambda^2}{\pi}$ of the wave front from which energy is drawn, is by hypothesis larger than the area S of the resonator; and we infer that the resonator acts, as far as its effect on the primary field is concerned, as a "sink" with a resultant convergence of the stream lines from the primary field to the mouth of the

resonator. This notion of the mechanism of selective absorption is useful not only in acoustics, but has an application in optics as well.

In addition to energy absorbed and reradiated, there is the inevitable reflection or scattering of primary energy by the resonator mounting, considered as a obstacle. In any practical case, the impedance of the yielding member (e.g., the diaphragm) of a sound detector will be very great, as compared with ρcS, and the conditions we have pictured above for a simple resonator would not obtain. We must consider the whole sound detecting instrument as a fixed sphere or disc, and to calculate the pressure on the diaphragm we must first determine the effect of the obstacle (i.e., the detector) on the sound field, from which investigation the velocity potential, and hence the pressure at any point on the surface of the detector can be found. The effect on the field of the yielding of the diaphragm is practically *nil*, but even so the problem offers great theoretical difficulty, and its solution cannot be attempted here.

For low frequencies, in the case of a spherical obstacle of radius a the total energy scattered in all directions is $\frac{7}{9}(ka)^4$ times the incident ènergy. The corresponding figure for a thin disc, is $\frac{16}{27}(ka)^4$, if a is the radius. (Lamb, § 81.) The proportion of energy scattered is thus small for small obstacles, but it is evident that it rises rapidly with the size of the obstacle, and more rapidly with frequency than the radiation resistance b_1, for example. As the frequency rises of course a greater proportion of energy is reflected (i.e., scattered) from the face of the obstacle toward the source until finally a condition of normal reflection is reached, at that surface. At very high frequencies the pressure on a stiff diaphragm facing the source is twice the excess pressure of the incident waves. [Cf. eq. (121).] In experimental work the only available recourse, to avoid the obstacle phenomena and the allowance that must be made for them when a very stiff detecting instrument is used, is to flange it with an infinite rigid wall, in which case (121) is substantially correct for *all* frequencies. If on the other hand it is possible to

use a light tuned detecting system, whose resistance (including that due to radiation) is about equal to the radiation resistance of the medium, good results will be obtained with little distortion of the primary field.

Those who wish to pursue the relatively complicated calculations necessary to deal rigorously with obstacles, sound shadows, and the scattering of radiation are referred to Lamb (§§ 79, 80, 81) and to Rayleigh (Chap. XVII).

We may finally consider how a resonator, when placed near a source of sound, can be used to enhance sound vibrations. According to (258) if kx is a small quantity the strength of the source is effectively raised by virtue of the presence of the resonator, in the ratio $1 : kx$; the intensity at some distant point will therefore be greater in the ratio $1 : k^2x^2$. This seems paradoxical, since if the resonator were distant from the source it would only give out what energy it could abstract from a certain limited part of the wave system due to source (1). But it must be recognized that if source (1) is, as we have assumed, a point source, the principal component of the velocity is 90° out of phase with the pressure there [cf. equations (161) and (161*a*), § 36] so that the point source is not of itself a very efficient radiator. The introduction of the resonator *near the source* provides a means whereby the excess pressure can produce a large velocity with which it is in phase, with a consequent gain in radiating power. That is to say, by decreasing the radiation resistance R, and at the same time compensating for the inertia reactance of the source itself (through tuning the resonator) we can make very large the quantity $\frac{p^2}{R}$, which is the radiated power.

48. *Acoustic Radiation Pressure*

In dealing with the impact of sound waves on a wall, it may have occurred to the reader that here (as in the electrodynamic case) there must be a positive pressure due to radiation. This is indeed true, but it is a second order effect, which we omitted

for that reason in discussing plane waves in § 31. The matter has been considered by Rayleigh in two papers published since [1] the second edition of the "Theory of Sound," and is not without practical application in measurements of sound intensity. Rayleigh's method for finding the force required to confine the vibrations at the end of a string is to be illustrated in Problem 39, which will give the reader an idea of the mechanism of radiation pressure. This pressure depends primarily on the *potential* energy residing in the medium, and is equal to the *mean energy density* associated with the wave motion.

To give Rayleigh's proof for the acoustic case would require the use of a general integral of the hydrodynamical equations of motion which we have not dealt with; but we shall make use of a simple and elegant treatment due to Larmor [2] for the general case of plane waves of any sort, normally reflected from a wall. The wall is free to move, normal to itself, and is pushed with uniform velocity v to meet the advancing wave train, of wave velocity c, and mean energy density E. If the wall were stationary, the total energy density due to incident and reflected wave trains would be $2E$. The length of the wave train incident on the moving wall in unit time is $c + v$, and in transmission, owing to the continuous encroachment of the wall, this is compressed into a space of length $c - v$. The energy density in the reflected wave is thus increased in the ratio

$$\frac{E + \Delta E}{E} = \frac{c + v}{c - v} = 1 + \frac{2v}{c}, \tag{260}$$

since v is to be taken as a vanishingly small quantity. We therefore have $\Delta E = \dfrac{E \cdot 2v}{c}$, and the total energy density is now $2E + \Delta E$. The increase in the total energy in the region of length c before the moving wall is $\Delta E \cdot c$ and this can be accounted for only on the basis of the work done by the wall in

[1] *Phil. Mag.*, III, 1902, p. 338, and X, 1905, p. 364; both reprinted in Vol. V of his "Scientific Papers." A review of the matter is given by W. Weaver, *Phys. Rev.*, 15, 1920, p. 399.

[2] *Encyc. Brit.*, Vol. 22, article on "Radiation."

compressing the radiation. If Ψ_R is the mean radiation pressure, the work done by the wall in unit time is $\Psi_R \cdot v$, hence

$$\Psi_R \cdot v = \Delta E \cdot c = 2E \cdot v, \tag{261}$$

that is, the mean radiation pressure is equal to the mean energy density in the medium adjacent to the wall.

The intensity of the radiation is equal to the product of the Energy Density and the wave velocity (§ 31), hence radiation pressure can be made the basis of sound intensity measurements if a sufficiently sensitive radiometer is available. This has actually been accomplished in experiments with high frequency under-water waves.[1] The radiometer is a simple torsion balance; a thin hollow metal box filled with air, on being submerged serves as a very good totally reflecting vane. The apparatus is simple and easily calibrated, and is an excellent example of the technique of modern experimental acoustics.[2]

PROBLEMS

31. In the sandwich-like high-frequency radiator described on p. 142 (§ 41) show that the two iron slabs must be one-quarter wave length thick for resonance, assuming that the inside crystal layer drives the adjacent boundary of the iron surface with a prescribed velocity.

32. A telephone diaphragm (clamped circular plate) has a total mass of 5 grams, a radius of 3 cm., a natural frequency of 1000, and a damping coefficient of 200 in air. It is placed in water, with one side only exposed to the fluid, and care is taken to prevent change in the equilibrium position of the diaphragm due to hydrostatic pressure. Find approximate values of the natural frequency and damping of the system under these new conditions.

[1] Such a radiometer was designed and used by A. P. Wills for measuring the radiation from the high-frequency vibrator previously described (§41). A similar instrument was used by Langevin; see the article on "Echo Sounding" previously cited.

[2] The ponderomotive effect of one vibrating system on another in the same sound field may be noted here. This has been investigated recently by E. Meyer (*Ann. d. Phys.*, 71, 1923, p. 567), and is briefly discussed by E. Waetzmann (*Phys. Zeit.*, XXVI, 1925, pp. 746–747).

33. Assuming that the diameter of the end of a tube is small as compared with the wave length, what is the reflection coefficient for the energy when a wave within the tube reaches the open end?

34. A cylindrical tube of unit radius and length one metre is driven at one end, the other end being open to the air. What is the driving point impedance of the tube, for low frequencies?

35. Obtain the typical equations for the phenomena of transmission and the production of standing waves in cylindrical tubes, by a suitable modification of the theory of the exponential horn.

36. For an exponential horn, find the phase relations between pressure and velocity for frequencies below the critical frequency.

37. Derive a general expression for the behavior of a resonator in a sound field, assuming that it is not tuned to the driving frequency, but that the wave length is large as compared with the resonator.

38. In §43 we have seen that an inertia reactance at the end of a tube is equivalent to an increase in length of the tube. It is likewise true that the stiffness reactance due to a bulb at the end of a tube, is equivalent to a decrease in the length of the tube. On these principles calculate the natural frequencies of a tube of length l, and section S, closed at one end by a rigid piston, and at the other end by a bulb of capacity V_0. (Rayleigh, II, § 317.)

39. A long string of tension τ passes through a heavy sleeve, free to move along the string, at a point $x = 0$ near one end. A transverse wave whose displacement is $\xi_0 \sin (\omega t + kx)$ travels along the string to the sleeve; it is totally reflected there, as all transverse motion of the sleeve is prevented. Show that the mean (space) energy density in the string is $\tau k^2 \xi_0^2$; and by a resolution of forces due to tension in the string at the point $x = 0$ show that the mean (time) force tending to displace the sleeve along the string is equal to $\tau k^2 \xi_0^2$, that is, the mean energy density in the string.

40. A plane wave is normally incident on a rigid infinite wall, in which there is a circular opening whose conductivity K is small as compared with the wave length. Show that the rate of flow of energy through the hole is to the intensity in the original wave as $2K^2 : \pi$, approximately (cf. Lamb, § 82). Why are you justified in neglecting the radiation resistance in calculating the velocity in the orifice?

CHAPTER V

The Acoustics of Closed Spaces; Absorption, Reflection and Reverberation

50. *Architectural Acoustics*

Most acoustical experiences take place in spaces bounded by reflecting and absorbing walls which tend to confine the sound energy and give rise to standing waves and other complications which do not occur in an unlimited medium. On account of their practical importance, considerable study has been devoted to these effects and the theory of Architectural Acoustics has been elaborated to deal with them. This is to be the main concern in the present chapter.

Practically, Architectural Acoustics deals with the phenomena which occur when speech or music are produced before an audience in an assembly-room; consequently we must recognize first of all what quality it is in sounds of this character which lies at the bottom of the difficulty of correct transmission. This quality depends on the fact that in speech and music the component sounds, as originally produced, consist of wave-trains of definite lengths, which overlap in a certain prescribed way, and unless these are reproduced at the ear of the listener with all the original sequence of phenomena (amplitude as a function of time) he hears a distorted and often very unsatisfactory version of the original composition. These distortions are really two-fold, though both effects depend on multiple reflection. In the first place it takes time for any source to establish radiation of a given mean energy density in an enclosure; and conversely, it takes time for the mean energy density to decay to zero, when the driving force is removed. This of itself will introduce a time distortion of any individual tone-component of limited duration. This effect throughout the

whole enclosure, particularly the hang-over at the end of the interval, is known as *reverberation*; it is analogous (but not equivalent) to the persistence of the natural oscillations in a system of several degrees of freedom, until all the energy is drained away. Secondly, owing to the recurrent reflection, there is produced in the enclosure a separate pattern or distribution of standing waves for each particular frequency of tone; this is a kind of distortion in space, and at any point in the enclosure, at a given instant, the resultant vibration due to all these interference patterns is not likely to be a sound of the same energy-frequency distribution as the original. To distinguish this latter effect from reverberation we shall call it *local wave-distortion*. To summarize then, there is imposed on the listener a confusion of two related effects, (a) reverberation, or the failure of the mean energy density to respond faithfully to the power variation at the source, and (b) a local effect of wave distortion, varying in both space and time, due to the idiosyncrasies of the standing wave system.

It might be thought off-hand that the universal cure for these effects would be to provide sufficient damping or absorbing power at the boundaries of the enclosure to eliminate standing waves entirely, or in other words, to simulate the transmission conditions prevailing in open space. Band music for example sounds mellow and agreeable in the open, if there are no disturbing noises present. But here enter psychological or rather esthetic considerations: music and oratory depend for some of their effects on sonority, and in these instances reverberation is of advantage in prolonging or enhancing certain tones in order to attain an artistic result. Reverberation also increases the overlapping of one tone on another, on the time scale, an effect which is sometimes desired. In addition there is usually some limitation on the amount of power which can or which should be expended at the source. In the case of a brass band playing in the open, the character of the music and the amount of available power usually suffice to satisfy the sense of loudness of a large body of listeners, but an equally agreeable effect would not be obtained with a string

quartette playing chamber music, or with a vocal soloist, because of the too rapid dissipation of energy from the listening area, in relation to a necessarily restricted power output at the source. In these latter cases, it is necessary, in order to fully appreciate the performance, to enclose the radiation and make use of reverberation to increase the mean energy density in the audience hall. All this implies, of course, that reverberation must be under strict control, and we may remark that in such purely physical investigations as sound analysis, etc., the distortions due to reverberation are not to be tolerated.

It is now clear that to solve any problem of Architectural Acoustics a delicate adjustment is involved; first we must know just how much reverberation is demanded by the performance given (which is a matter of taste), and then we must determine what physical measures are best adapted to bring about this result in the given enclosure, which is a matter of applied science. Here we have the whole basis [1] of Sabine's work; he diligently collected data on which judgment could be based as to the degree of reverberation desired in typical performances, and he made a thoroughgoing investigation of the physical factors which control reverberation. In what follows we shall take for granted the judgment of qualified critics as to what is optimum reverberation, and confine ourselves principally to the purely physical aspects of the problem.

The subject of Architectural Acoutsics is sufficiently broad to include in addition many special effects, such as echoes, focal properties of reflecting surfaces, whispering galleries, etc. Most of these, though possessing features of interest, are not novel; many such cases have been previously treated, and these we shall leave mostly to one side, with references to Rayleigh (II, Chap. XIV) and to Sabine, where interesting discussions of these phenomena will be found.[2]

Before proceeding to the theories of this chapter we must mention the limitation which is imposed on listening con-

[1] Wallace Clement Sabine, *Proc. Am. Acad.*, XLII, June 1906. Reprinted in "Collected Papers on Acoustics," p. 69.

[2] Sabine, "Papers," p. 255; "On Whispering Galleries."

ditions whenever there are disturbing noises present. Let us suppose that the energy of the sounds to which we are paying particular attention fluctuates within a certain range; then if the energy level[1] of the interfering noises encroaches on the lower end of this range, clearly all the energy levels in the range of the sounds for which we are listening must be raised in order that they may stand out properly against the background of noise. If this cannot be done (either for mechanical or esthetic reasons) then the quality of the performance is impaired. This problem of the preservation of energy levels, both in themselves, and sufficiently above the energy level of disturbing noises, arises in all cases of sound reception; it is too technical for further discussion here, but the reader must understand the reservation which is always implied, regarding absence of disturbing noises, in the theories which follow. For an example of work in this field, see V. O. Knudsen, "Interfering Effect of Tones and Noise on Speech Reception," *Phys. Rev.*, 26, 1925, p. 133. After giving due attention to the mechanics of reverberation, in sections following, the reader should find of interest the series by P. E. Sabine on "The Nature and Reduction of Office Noises," in *Am. Arch.*, 121, 1922, p. 441, p. 487, and p. 527.

51. *Reflection and Absorption*

Since reflection is the primary agent in confining radiation within an enclosure it is important to gain an idea of the mechanism of this effect, particularly at the surface of an absorbing wall. Wall surfaces vary in structure from hard glazed tile and bricks which are nearly impervious to sound waves, to coverings of fabrics which are light and porous and which therefore transmit and absorb incident sound radiation very well. In between these classifications are a variety of materials; wood, plaster, paper, rough bricks, etc., all of which

[1] The *energy level* is a somewhat loose but convenient term to indicate the energy density or the power at the listening point, measured from any convenient zero. The scale of Transmission Units (suggested on p. 76) is usually employed to designate differences in level.

are porous to various degrees. It is not necessary here to deal with any particular material, but by a further development of the ideas previously applied to account for sound absorption (§ 34) we can arrive at a sufficiently good understanding of the reflection and absorption phenomena at an average wall surface.

In § 34 we dealt with the classical conception (due to Rayleigh) of a wall surface of light material which was honeycombed with small channels and crevices leading inward from the exposed surface. The incident sound waves penetrate to a certain degree into these narrow conduits where they are extinguished by frictional resistance. (The theory of this effect is given in Appendix A, the results of which should be familiar to the reader before proceeding with the following discussion.) In the present instance we shall assume that the walls of the channels in the absorbing material are hard and unyielding, also infinitely thin; so that the channels are closely packed in a hexagonal honeycomb structure, and each one can be regarded as a small cylinder of circular section, without doing violence to the argument. We can easily find the impedance presented by the mouth of one of these conduits, and hence the driving point impedance of the wall to sound waves; and by a comparison of this impedance with the radiation resistance of the free medium containing the incident and reflected sound waves we can calculate how much of the incident radiation is reflected, and how much is transmitted only to be absorbed.

In Appendix A it is shown that the particle velocity is propagated in a very narrow tube according to the equation:

$$\dot{\xi} = \dot{\xi}_0 e^{-\beta x} e^{i(\omega t - \beta x)}, \qquad (B')$$

in which

$$\beta = \frac{1}{c}\sqrt{\frac{\omega R_1}{2\rho}}, \qquad R_1 = \frac{8\mu}{r_0^2}. \qquad (K')$$

This implies for the excess pressure a similar equation

$$p = p_0 e^{-\beta x} e^{i(\omega t - \beta x)}, \qquad (262)$$

from which we derive the presure gradient

$$\frac{\partial p}{\partial x} = -\beta(1 + i)p. \tag{263}$$

Making use of the familiar relation between the pressure gradient, the velocity of flow, and the resistance coefficient R_1 appropriate to the conduit, we have:

$$\dot{\xi} = -\frac{1}{R_1}\frac{\partial p}{\partial x} = \frac{\beta(1 + i)}{R_1}p = \frac{1}{Z}p, \tag{264}$$

in which

$$Z = \frac{R_1}{\beta(1 + i)} = \frac{R_1(1 - i)}{2\beta}, \tag{265}$$

is the impedance of the conduit, per unit area of section. From (K') (in which inertia effects are neglected) we have, according to Poiseuille's law, $R_1 = \frac{8\mu}{r_0^2}$; hence, substituting also for β, the impedance is:

$$Z = \frac{c(1 - i)}{2}\sqrt{\frac{8\mu}{r_0^2}\cdot\frac{2\rho}{\omega}} = \frac{2\rho c(1 - i)}{r}\sqrt{\frac{\mu}{\rho\omega}} = \rho c M(1 - i). \tag{265a}$$

M is a convenient factor containing the kinematic viscosity $\left(\frac{\mu}{\rho}\right)$, the frequency, and the radius of the conduit.[1]

[1] Inasmuch as we have taken $\kappa' = \gamma p_0$ for the bulk modulus of the medium in the conduit (see Appendix A) the constant c is of course the unmodified velocity of sound in the free medium. In the discussion of this problem Rayleigh prefers for very small tubes the isothermal bulk modulus, $\kappa = p_0$, which of course leads to the Newtonian velocity of sound, $\frac{c}{\sqrt{\gamma}}$. The difference is not material to the argument here; if we wished to use the Newtonian velocity, still retaining the convenient factor ρc in (265a), we might substitute for M the quantity $M' = \frac{M}{\sqrt{\gamma}}$. It must be particularly noted that *neither* the Newtonian nor the ordinary Laplacian value of the velocity is the phase velocity of the waves which diffuse into a narrow tube; the latter velocity is very much smaller, as shown in Appendix A. Incidentally we remark that for a conduit of diameter .02 cm. (as used in the example which is to follow) it is not certain which is preferable for the bulk modulus—the adiabatic value, γp_0, or the isothermal value, p_0.

Inasmuch as Z differs from ρc, the radiation resistance of the adjacent free medium, there will be reflection from the face of the wall. Rewriting eq. (120), § 31, noting in that equation that $R_2 = Z$ and $R_1 = \rho c$, we have for the particle velocity of the reflected wave, in terms of that of a normally incident wave:

$$-\frac{\dot{\xi}_0'}{\dot{\xi}_0} = \frac{Z - \rho c}{Z + \rho c} = \frac{M(1 - i) - 1}{M(1 - i) + 1}. \tag{266}$$

The reflection coefficient for the incident energy, obtained by squaring this ratio and rationalizing the result is:

$$\boldsymbol{R} = \frac{(M - 1 - iM)^2}{(M + 1 - iM)^2} = \frac{2M^2 - 2M + 1}{2M^2 + 2M + 1}e^{2i\phi}, \tag{267}$$

in which ϕ is the phase difference between $\dot{\xi}_0'$ and $\dot{\xi}_0$, a quantity with which we are not immediately concerned.

It can easily be shown that $|\boldsymbol{R}|$ has a minimum value for $M = \frac{1}{\sqrt{2}}$. This means that for any given honeycomb structure made up of conduits of capillary dimensions there is a frequency of minimum reflection, or maximum absorption. (The bearing of this result will presently appear.) For example, suppose that in the wall structure the conduits are 0.2 mm. in diameter, or $r_0 = 10^{-2}$. Then, since $\frac{\mu}{\rho} = .13$ (approximately, for air) we find that for $M = \frac{1}{\sqrt{2}}$, the corresponding frequency is $\frac{\omega}{2\pi} = 1600$, roughly. For this frequency we compute, according to (267), $|\boldsymbol{R}| = .17$ (the minimum reflection coefficient), and from this the fraction $1 - |\boldsymbol{R}| = .83$ of the incident energy which passes on into the wall structure where it is absorbed. Noting that M varies as $\frac{1}{\sqrt{\omega}}$ we arrive at the following series of values for $|\boldsymbol{R}|$ and $1 - |\boldsymbol{R}|$ which illustrate graphically the theoretical behavior of this type of wall toward incident sound waves in air:

Theoretical Reflection and Absorption Coefficients for a Wall of Closely-Packed Honeycomb Structure

Diameter of Conduits, .02 cm.

Frequency, f	Impedance Factor, M	Reflection Coefficient, $\|R\|$	Absorption Coefficient $1-\|R\|$
200	2	.38	.62
400	$\sqrt{2}$	.28	.72
800	1	.20	.80
1600	$\frac{1}{\sqrt{2}}$	.17 (min.)	.83 (max.)
3200	$\frac{1}{2}$	.20	.80
6400	$\frac{1}{2\sqrt{2}}$	.28	.72

Note first that since M varies as $\frac{1}{r_0\sqrt{\omega}}$, these corresponding values of M, $|R|$ and $1 - |R|$ will all be shifted to lower frequencies if r_0 is made greater than .01 cm., that is, if the wall is made of coarser texture. If r_0 is taken for example, equal to $.01\sqrt{2}$, than the frequency corresponding to the quantities in a given line of the table will be lowered by one octave.

If the absorption coefficient $1 - |R|$ is plotted against frequency, a very good resonance curve is apparently obtained. This resemblance is evidently accidental, as no resonance phenomenon, or selective absorption (of the type described in § 47) has been implied in the problem we are considering. But it seems reasonable to infer from the results obtained on this simplified theory of the mechanism of sound absorption[1] by

[1] The essential soundness of the classical method of dealing with porosity, which we have followed, is attested by an observation of P. E. Sabine (*Phys. Rev.* XVII, 1921, p. 379). "Experiment indicated that the change in absorption with change in thickness [for hair felt] follows the same law as the reduction of energy flow of an air stream through the material. The dissipation of energy of flow through the material at constant pressure approaches a limiting value as the thickness of the material is increased." In other words, the mechanism of dissipation in narrow conduits is essentially the same, in both static and dynamic cases.

porous bodies, that we can never expect to find a homogeneous material which will reflect and absorb sound waves according to coefficients which are independent of frequency. The reader is warned, however, not to confuse the selectivity due to porosity alone, with selective absorption based on resonance in layers of absorbing material. This latter effect will be discussed more fully in § 52. The two effects are of course superimposed in the case of any yielding material such as felt.

The particular close-packed honeycomb structure we have been studying has a high absorption coefficient, as all of its area is effective for transmission purposes. But we can easily imitate a harder or less permeable wall by filling up some of the conduits with infinitely rigid material, taking care of course to distribute the filled-up spaces so uniformly, that in any reasonably small area, however chosen, the proportion of perforated to unperforated surface is always the same. The effect produced is merely to increase the impedance of the wall. If a is the part of unit area which is now of open conduits, and b the unperforated part, then before, with both a and b open, the impedence due to a and b in parallel was Z per unit area, or adding admittances, we had:

$$\frac{1}{Z} = \frac{a}{Z} + \frac{b}{Z}, \qquad \text{since } a + b = 1.$$

Now, with the impedance of the part b infinite, the impedance per unit area of the wall is no longer Z, but $Z' \equiv \frac{Z}{a}$; and a can be made any fraction less than unity. Putting Z' for Z in (266) and (267) we have for the reflection coefficient of the modified wall:

$$|R| = \frac{2M^2 - 2Ma + a^2}{2M^2 + 2Ma + a^2}. \tag{267a}$$

(An application of the equation of continuity to the flux in the neighborhood of the openings (Rayleigh, II, § 351) leads to the same result.) From (267a) we observe that for a given value of M (that is for constant impedance per conduit), the reflecting

power of the wall increases as a becomes smaller; $|\boldsymbol{R}|$ finally becomes equal to unity when a vanishes, that is, when all the conduits are rigidly closed.

To this point we have dealt, in abstract terms, with reflection and absorption primarily for the purpose of understanding them. In practice a more empirical method of dealing with these properties is sufficient, and at the same time necessary, because what interests the practitioner of Architectural Acoustics is the application of materials of known gross absorption and reflection coefficients to control reverberation. One method of measuring the absorbing power of a material has already been described (cf. Wente's Experiment, § 34), and older methods are available, as for example the one due to Sabine, which depends on the relation between the reverberation, the dimensions of the enclosure, and the total absorbing power of the bounding surface. Sabine very logically took as his standard absorption coefficient, that of the free medium, or more precisely, an open window of area one square meter.[1] The absorbing powers of everything else, e.g. walls (per square meter), furniture (per piece), the audience (per person), etc. were all fixed by his experiments, in terms of this standard. (A great many absorption data are given in his "Collected Papers," e.g., "Absorbing Power of an Audience, etc." pp. 52–60.) The utility of data of this sort will appear when we come to consider reverberation.

[1] P. E. Sabine (*Phys. Rev.*, XIX, 1922, p. 402) has called attention to the fact that, due to diffraction, the shape as well as the area of an absorbing surface, or a window opening, will affect the absorbing power of a given area. Some reflection from an open window also takes place. It appears from P. E. Sabine's measurements that the smaller and narrower the opening, the greater the transmission coefficient per unit area.

W. C. Sabine's earlier measurements were made by a substitution method, using pew cushions as a standard of comparison, then comparing these with open window space, assuming that the latter transmits perfectly, and in proportion to its area. In his later work, however, he determined the absorption coefficient of a given material by the introduction of a known area of material. The total absorbing power of the room was determined by the "four organ pipe experiment," ("Collected Papers," p. 35).

Practically all the coefficients used are those obtained by the later method, through Prof. Sabine does not note the change of method in any of his papers. (Comment of P. E. Sabine, by letter, Jan. 30, 1926.)

52. *Layers of Absorbing Material*

In applying a layer of absorbing material (such as an inch of felt) to a hard wall surface, if the sound waves are not extinguished in one transit back and forth through the material, it becomes necessary to take into account the multiple reflections in the layer, and the consequent emergence of sound waves from within the layer which will increase its reflecting power. Of course by actual measurement the gross absorption and reflection coefficients of the structure can be found, and we have observed that these data would be sufficient for engineering purposes; but it may be desired to relate them to the more fundamental constants which are based on the transmission of sound waves in the absorbing material.

In general, there are two methods of attacking the problem; these we may refer to as the impedance and the wave methods. Both are empirical; to apply either method we assume that sound waves are propagated in the material in accordance with the familiar equation[1]

$$\dot{\xi} = \dot{\xi}_0 e^{-(\alpha + i\beta)x} e^{i\omega t}, \tag{268}$$

in which α and β (the attenuation and phase factors) are to be experimentally determined. The factor β is established if the phase velocity $c' = \dfrac{\omega}{\beta}$ is known. It is also necessary to know either the density (ρ') or the radiation resistance $R_2 \equiv \rho' c'$; the quantity c' in this expression being approxi-

[1] It has been established by P. E. Sabine (*Phys. Rev.* XVII, 1921, p. 378) for a porous material (hair felt) that "the logarithm of the reduction in intensity of the transmitted sound is proportional to the thickness of the material through which it passes" which is in accordance with the assumption of eq. (268). The attenuation coefficients as measured by P. E. Sabine vary with frequency, in much the same way as the typical curves for felt obtained by W. C. Sabine, to be referred to as the end of § 52. (See also P. E. Sabine, "Transmission of Sound through Flexible Materials," *Am. Arch.*, Sept. 28, p. 215 and Oct. 12, 1921, p. 266.)

The work of P. E. Sabine on absorbing properties of materials, and other problems in Architectural Acoustics was done, and is being continued at the Wallace Clement Sabine Laboratory, Riverbank, Geneva, Illinois. The plan and the activities of this Laboratory (which is a memorial to the late W. C. Sabine), are described in *Am. Arch.*, 116, 1919, p. 133–138.

mately equal to the phase velocity. (This is on the assumption that α^2 is not comparable with β^2; we are now dealing with a yielding material whose stiffness, density and dissipation constants differ from those of the pure air in a narrow conduit, for which we found $\alpha = \beta$ in the preceding section. In the present case, on account of the inertia of the material, c' will be less than the velocity of sound in the adjacent gaseous medium; c' will be further lowered by a second order quantity dependent on α, and α itself varies as the ratio of dissipation to density in the material. As a result of all these effects, we should assume that $\beta > \alpha$ except at very low frequencies. Strictly speaking $R_2 \equiv \rho' c'$ is a complex quantity; but here we take c' as the approximate phase velocity, which is a real quantity.) If we know these fundamental constants we can estimate the sound absorbing and reflecting characteristics of the given material, in any configuration. The density can of course be checked by weighing; but because of the composite structure of absorbing materials, and the variation of their properties with the closeness of packing, it is usually not feasible to obtain precise results for the other quantities.

If the properties of the material are known, the impedance method is best adapted to determine the reflecting power of a finite layer. To illustrate, if the layer is applied to a rigid wall surface, the problem is similar to that of the tube closed at one end, which was solved in § 32 assuming that there was no dissipation in the medium. There we found for the driving point impedance of the tube, of length l,

$$Z_0 = - i\rho c \cot \beta l. \qquad (136b)$$

This result can be applied to the problem of an absorbing layer if we substitute for ρc the radiation resistance R_2, characteristic of the material, and for β, the corresponding quantity $\beta - i\alpha$ which takes account of dissipation, in the present problem. This equation may now be written:

$$Z_0' = R_2 \frac{\cos (\beta - i\alpha)l}{i \sin (\beta - i\alpha)l} = R_2 \coth (\alpha + i\beta)l \qquad (269)$$

and determines, at least theoretically, the impedance of the layer. A comparison of this impedance with that of the adjacent free medium, making use of (120) as before, gives the (amplitude) reflection coefficient for the layer, in both absolute value and phase.

If on the other hand the constants of the material are not known, and it is desired to measure them, what we shall call the wave method has certain advantages in practice. The difference between the two methods is only in the point of view; both lead to the same results, but it may be of interest to verify this conclusion.

We shall suppose that by some scheme such as Wente's (§ 34) we have measured the gross reflection coefficient of a layer backed by a rigid wall, and that we wish to determine the true reflection coefficient, that is the reflection coefficient at the boundary of an infinitely thick layer. Let r equal this latter quantity; then for unit amplitude of the incident wave, the amplitude of the entering wave is, by eq. (120a), § 31,

$$\xi_2 = t_{12} = \frac{2R_1}{R_1 + R_2} = \frac{2}{1 + R}, \tag{120a}$$

if R_2 is the radiation resistance of the absorbing material, and R its relative radiation resistance with respect to the other medium. On the same basis we have for the reflection coefficient

$$r = -\frac{R_2 - R_1}{R_2 + R_1} = \frac{1 - R}{1 + R}, \tag{120}$$

to which we shall add the relations previously obtained

$$t_{21} = \frac{2R_2}{R_1 + R_2} = \frac{2R}{1 + R}, \qquad t_{21}t_{21} = 1 - r^2. \tag{120a}$$

In passing through the layer to the rigid wall, the amplitude t_{12} becomes $t_{12}e^{-(\alpha + i\beta)l}$, or more concisely, $t_{12}e^{-x}$. In reflection from the wall, there is a phase change of -1, and in transit back through the layer a phase change of e^{-x}; finally, in emerging from the layer at the boundary there is an amplitude reduc-

tion in the ratio $t_{21} : 1$. Combining all these effects, we have, for the first internally reflected wave, on emergence from the layer, the amplitude

$$\xi_2 = - t_{12} \cdot t_{21} \cdot e^{-2x} = - (1 - r^2)e^{-2x},$$

which is added to the amplitude $\xi_1 = r$ of the externally reflected wave. Proceeding to the second internally reflected wave, we note that it starts from the boundary with an amplitude $- t_{12}e^{-2x} \cdot (- r)$, and after undergoing transit through the layer, reflection at the wall, and emergence, its amplitude is

$$\xi_3 = + t_{12} \cdot t_{21} \cdot (- r)e^{-4x} = - r(1 - r^2)e^{-4x}.$$

We infer then that the gross amplitude of the wave reflected from the layer is the sum

$$\xi_r = \xi_1 + \xi_2 + \xi_3 + \ldots = r - (1 - r^2)e^{-2x} - r(1 - r^2)e^{-4x} \ldots \tag{270}$$

that is,

$$\xi_r = \Sigma\xi_j = r - \frac{(1 - r^2)e^{-2x}}{1 - re^{-2x}} = \frac{r - e^{-2x}}{1 - re^{-2x}}, \tag{270a}$$

summing the convergent geometric series which follows r. The advantage of this formula is that if ξ_r (the ratio of reflected to incident wave amplitude), is found for several different thicknesses of the material, the quantities r and $e^{-x} = e^{-(\alpha+i\beta)l}$ can be determined from the simultaneous equations containing the experimental data. In solving the equations certain approximations are necessary; these are best devised to fit the necessities of each given set of data. Theoretically, if three layers of different thicknesses are tested (as in Problem 41, below), we have a method of separating r, α, and β, the three principal constants of the absorbing medium. Knowing r is of course equivalent to knowing R_2, since R_1 is known; equation (270a) is easily obtained by substituting for R_2 in (120) the quantity Z_0' of (269).

To further illustrate the wave point of view, and a rough determination of the phase velocity for an actual material, we

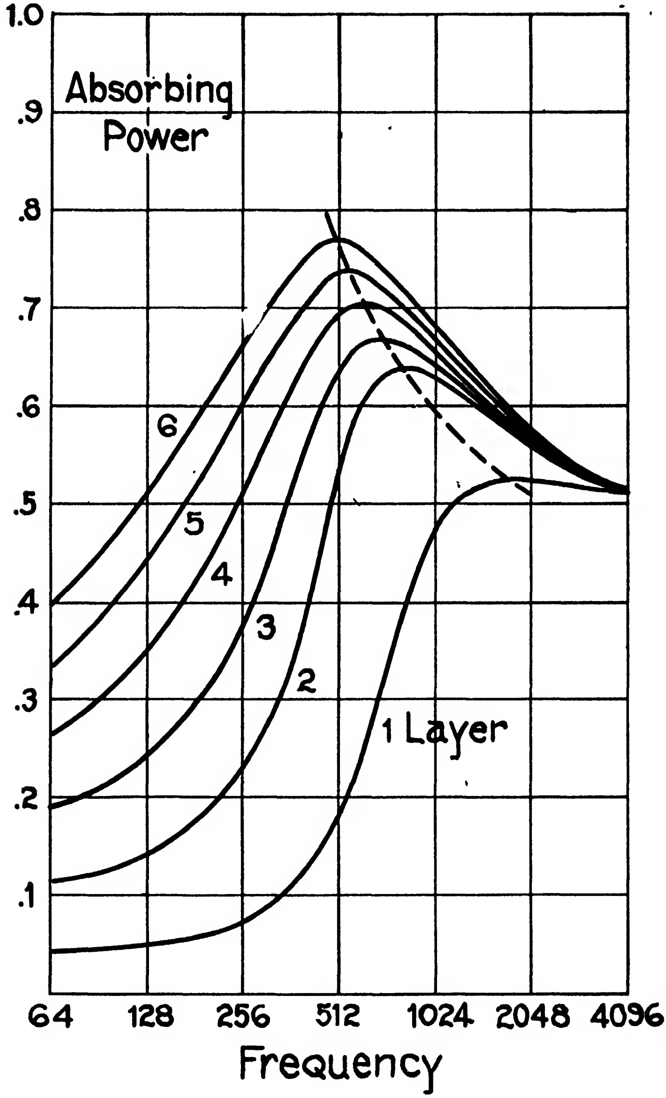

FIG. 20.—ENERGY ABSORBING POWER OF ONE KIND OF FELT, IN LAYERS OF THICKNESS INCREASING BY MULTIPLES OF 1.1 CM.

may consider some data taken by W. C. Sabine for a certain kind of felt. This was nominally of wool, and very porous. In Fig. 20 are shown the values of the (gross) absorption coefficients of layers of various thicknesses of this material.[1] The first layer was 1.1 cm. thick, the second 2.2 cm. thick, and so on in multiples of 1.1 cm. The *increase* of absorption with *thickness* is marked, particularly at lower frequencies; there is also noted a shift of the maximum absorption coefficient to lower frequencies as the thickness is increased. This shows, as pointed out by Sabine, a certain amount of resonance in the material; this reaction may be viewed as due to standing waves therein, if the thickness is sufficiently great, otherwise, to simple compression. If we ignore the fact that due to porosity alone, there will be a fixed frequency of maximum absorption, we may suppose that a layer of the material will ordinarily show resonance (and hence selective absorption) when it is one-quarter of a wave-length thick. From this we can determine roughly the phase velocity in the material, since $\lambda f = c'$. From the curves we take the frequencies of maximum absorption to be respectively 2048, 1024, 725, 610, 512, and 480, beginning with the thinnest layer. Multiplying each of these by four times the appropriate thickness, we get a series of values for c' ranging from $.91 \times 10^4$ to 1.27×10^4, with rising thickness, the mean being 1.01×10^4. This should be approximately the value of the phase velocity in the material; the (apparent) higher values of c' for the thick layers are what we should expect, as in these cases, the (apparent) frequency of maximum selective absorption is doubtless raised owing to the presence of a higher frequency of maximum absorption due to porosity alone. The effect of increased attenuation at the higher frequencies will also tend to lower the frequency of selective absorption in the thinner layers.

In Fig. 21 the same data are plotted as absorption coefficients in terms of thickness, each curve relating to a particular frequency. "Thus plotted, the curves show the necessary

[1] W. C. Sabine, *Proc. Am. Acad.*, XLII, June, 1906; or "Collected Papers," p. 99, Figs. 12 and 13.

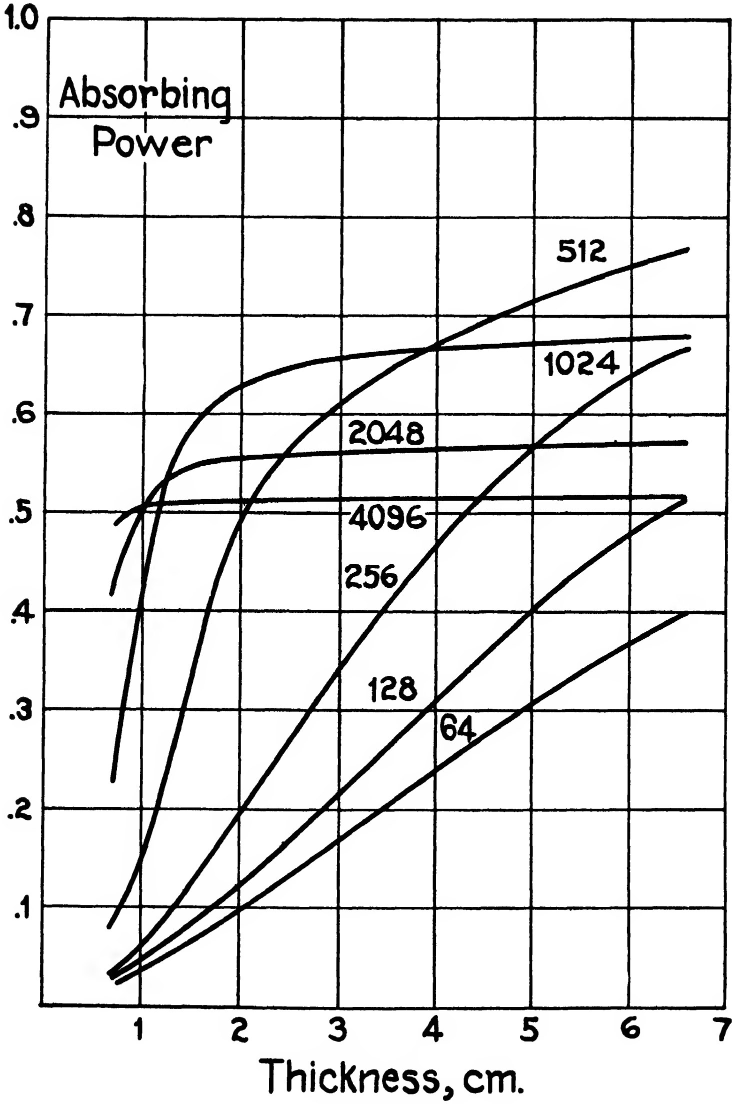

Fig. 21.—Data of Fig. 20 Replotted to Show, for Different Frequencies, Variation of Absorbing Power of Felt with Thickness of Layer.

thickness for practically maximum [1] absorption." It is evident from the curves (with one exception) that they tend to pass through the origin, that is, for zero thickness, the absorption would be negligible. This Sabine interprets as showing that there is no dissipative effect due to friction at the surface; we have omitted any such speculation heretofore, but it is interesting to note that this omission is justified. The main point of this family of curves is that they show much greater attenuation in the material for waves of higher frequency, two thicknesses being sufficient to bring about virtually maximum absorption for $f > 1024$. Finally, we may observe that the higher attentuation will enhance the reduction of the phase velocity at the higher frequencies, the effect of which reduction we have noted in the preceding paragraph.

In practice it may be necessary to deal with composite absorbing structures built up of strata of various materials. The methods we have illustrated are theoretically applicable in the study of all such cases, but the relations obtainable between the various quantities may be so cumbrous that they are of little service. Here experiment must be the deciding factor; but the procedure in many cases can be simplified if we use the theory we have given as a guide to the study of the component materials.

The measurement of the leakage of sound energy from enclosures (depending on transmission through windows, doors, partitions, cracks in walls, etc.), while not at first sight germane to this discussion of absorption, has its proper place in Architectural Acoustics, and should be mentioned. The point to be made is that such leakages are to be taken into account as additional dissipation from the enclosure, that is, as a virtual increase in the absorbing power of the bounding surface. The first experimental study of these effects was by W. C. Sabine [2];

[1] Cf. P. E. Sabine, *loc. cit.* "The values of the absorption coefficient [for hair felt] were found to increase with increasing thickness of absorbing material, but they approach limiting values which are reached at smaller thicknesses for the higher frequencies."

[2] *The Brickbuilder*, XXIV, Feb. 1915; or "Papers," p. 237: "The Insulation of Sound."

since that time much experimental testing has been done by P. E. Sabine [1] and F. R. Watson [2] in order to establish the correct transmission coefficients of doors, partitions, and other boundaries. Finally there is a very complete paper by E. Buckingham [3] which not only considers the theory of the experiments of sound transmission through walls, but gives a treatment of reverberation which will interest the reader, in connection with the theory of § 54 following.

53. *Reverberation in a Closed Tube*

A natural approach to the general theory of reverberation is through an analysis of the effects produced in a long tube of unit section, closed with absorbing walls at either end, and fitted at one end with a small source of sound. The source, radiating sound at a constant rate, sends wave trains of any desired length into the tube. The absorption of energy at the ends of the tube is proportional to the energy density in the tube, and since the energy density in the tube is constantly added to by the radiation from the source, there comes a time after starting the generator, when the radiated energy is in equilibrium with that absorbed, and a steady state prevails. The asymptotic growth of energy density toward this steady value is the first characteristic effect to be noted; and conversely, on stopping the generator, the energy in the enclosure decays exponentially to zero. Thus the time variation of the energy density in the enclosure does not follow faithfully that of the radiated energy, if the wave train is finite; and we have the characteristic phenomena of reverberation. The mean energy density is chosen as the best variable in terms of which to state the theory, no account being taken of frequency or phase.

[1] P. E. Sabine, *Am. Arch.*, July 28, 1920 "Transmission of Sound through Doors and Windows"; see also *ibid.*, Sept. 28 and Oct. 12, 1921, and July 4, 1923, the last paper dealing with transmission through walls; see also *Phys. Rev.*, 27, 1926, p. 116.

[2] F. R. Watson, *Univ. of Illinois Eng. Exp. Sta.*, Bulletin No. 127 (1922) "Sound Proof Partitions"; see also his "Acoustics of Buildings," New York, Wiley, 1923.

[3] *U. S. Bureau of Standards*, Sci. Paper No. 506, May 26, 1925.

It is assumed that the generator radiates a wave train of energy density E_1 into the tube; thus the rate of working of the source is cE_1 plus whatever power is lost locally due to the adjacent absorbing surface at that end of the tube. The length of the tube is l, and $\boldsymbol{A} \equiv 1 - \boldsymbol{R}$ is the (energy) absorption coefficient of the wall at either end, $\boldsymbol{R}$ being the (energy) reflection coefficient. The time required for n reflections of the original wave at the ends of the tube is a trifle greater than $\frac{nl}{c}$; the number of reflections per second is $m \equiv \frac{c}{l}$ and the mean free path between reflections is l. To find the energy density at any moment after the generator has started working we must add to the energy density E_1 of the original wave, the energy density of the once reflected wave ($\boldsymbol{R}E_1$), that of the twice reflected wave ($\boldsymbol{R}^2E_1$), and so on: hence just after the time $t = \frac{nl}{c}$ the energy density is

$$E_n = E_1(1 + \boldsymbol{R} + \boldsymbol{R}^2 + \ldots + \boldsymbol{R}^{n-1})$$

$$= \frac{E_1(1 - \boldsymbol{R}^n)}{1 - \boldsymbol{R}} = \frac{E_1}{\boldsymbol{A}}(1 - \boldsymbol{R}^n), \qquad (271)$$

taking the sum of the finite geometrical series. In terms of time this may be written

$$E_n = \frac{E_1}{\boldsymbol{A}}\left(1 - \boldsymbol{R}^{\frac{ct}{l}}\right) = \frac{E_1}{\boldsymbol{A}}\left(1 - e^{-\frac{\boldsymbol{A}'ct}{l}}\right), \qquad (271a)$$

taking $e^{-\boldsymbol{A}'} = \boldsymbol{R}$. The quantity $\boldsymbol{A}'$ is of the order of magnitude of $\boldsymbol{A}$; since $-\boldsymbol{A}' = \log \boldsymbol{R} = \log(1 - \boldsymbol{A})$, we have, on expansion,

$$\boldsymbol{A}' = \boldsymbol{A} + \frac{\boldsymbol{A}^2}{2} + \frac{\boldsymbol{A}^3}{3} + \ldots. \qquad (272)$$

From (271a) we note that the *rate* of growth of energy density varies *directly* as $\boldsymbol{A}'$, but the maximum energy density *finally attained* in the steady state varies *inversely* as $\boldsymbol{A}$. In other

words, a large absorption coefficient for the boundary surface makes the enclosure more responsive to power variations at the source, at the expense of a lessened maximum energy density or loudness.

The growth of energy density, for constant rate of working of the source, is analogous to the growth of charge density in a leaky condenser, which is charged by a *constant* direct current. This takes place according to the equation:

$$\frac{Q}{C} = IR\left(1 - e^{-\frac{t}{CR}}\right), \tag{273}$$

in which I is the constant charging current, C the capacity and R the resistance of the leak between the plates of the condenser. $\frac{Q}{C}$, the charge density, is analogous to the energy density E. The charging current I is analogous to E_1 since both measure rates. The conductivity $\frac{1}{R}$ is analogous to the absorbing power A, and the capacity C corresponds to the volume l of the tube; thus the analogy is complete.

Now suppose the sound generator is stopped, the energy density in the tube having attained the value $\frac{E_1}{A}$. The number of reflections in unit time is $m = \frac{c}{l}$; and as the waves pass back and forth between the absorbing surfaces an amount of energy $A \cdot E$ is absorbed at each reflection. The rate of change of energy density in the tube is therefore

$$\frac{dE}{dt} = -\,mAE, \tag{274}$$

and the solution of this gives, consistently with the boundary condition,

$$E = \frac{E_1}{A} e^{-\frac{c}{l}At}, \tag{274a}$$

for the decay of energy density. The electrical analogue is the same as before, the equation for the discharge of the leaky condenser being

$$\frac{Q}{C} = IRe^{-\frac{t}{CR}}. \tag{275}$$

A very real idea of what underlies the colloquial phrases "filling a room with sound" and "deadening the room" (to diminish reverberation) can be gained if we adopt another analogy which happens to be virtually ready made. We have found (§ 48) that the energy density of radiation is equivalent to its radiation pressure, a static effect. Hence for the immediate purpose we may substitute for the contained radiation a compressible fluid whose density and pressure are proportional to the density of the radiation. Fluid is introduced at constant rate by a source; the pressure within increases until the rate of dissipation of fluid through leaks or absorption is equal to the rate of supply; and finally if the supply is shut off, the pressure diminishes at a rate which is proportional to the product of the pressure and the conductivity of the leaks, or the overall absorption coefficient of the boundary. These are the essential ideas involved in the mechanism of reverberation, and may be applied, with some modification of detail, to an enclosure of any size, which has reasonably simple proportions.

If we define T, the *reverberation time* of the given enclosure as the time required for the energy density to sink from one prescribed level to another, then according to (274*a*), if any two of the quantities T, A or l are known, the other can be calculated. Let subscripts 1 and 2 refer to the two energy levels; then if $E_1 > E_2$, we have immediately

$$\frac{E_2}{E_1} = e^{-\frac{c}{l}AT} \quad \text{or} \quad AT = \frac{l}{c}\log\frac{E_1}{E_2}. \tag{276}$$

This equation states what might be called the *first law* of Architectural Acoustics; it will be sufficiently generalized in § 54 to cover all practical cases, in the form $AT = KV$, V being the volume of the enclosure and K a constant. What we wish to

discuss here is the determination of A in terms of T, if l (or V) is known. W. C. Sabine made a great many laborious experiments using the threshold (i.e. minimum audible) energy density for E_2, and for E_1 and energy level 10^6 as great. By varying the absorbing power of a given room, i.e. by adding or subtracting absorbing surfaces on the floor or on the walls, he was enabled, through the relation $A_1T_1 = A_2T_2$, to measure the absorbing powers of all the usual materials, objects, etc., in terms of the open window of unit area for which he took $A = 1$. Since it was impracticable to line a room of any considerable dimensions throughout with material of uniform absorbing qualities, many differential experiments had to be made, in order to accumulate a sufficient variety of absorption data for working purposes; but when the data were finally accumulated, it was found that the process could be reversed, and by calculating the total absorbing power of the boundary surfaces, and the objects contained in the room, a sufficiently accurate prediction of the reverberation time could be made.

The essential soundness of the reverberation method of measuring absorption coefficients is not to be questioned, but owing to the dependence on the ear for determining the time at which the threshold intensity level was reached, repeated observations had to be made in order to attain the required degree of precision in the results. (Sabine customarily took 10 or 20 observations in any given case.) A chronograph was necessary to register time intervals; and there were other cumbersome features of his experiments, such as the use of the constant pressure air reservoir (for actuating the organ pipes used as sound sources) which entailed considerable manipulation. The organ pipes had to be calibrated, in connection with the air supply, in a physical laboratory. Taken altogether, the work was difficult and tedious, as compared with present laboratory practice, and if repeated today, would doubtless be greatly facilitated by the use of better instruments which are now available.

It seems, however, from the practical standpoint, that small scale methods based on the mechanics of sound in tubes,

and independent of reverberation, offer definite advantages in the determination of the absorption and reflection coefficients at the surfaces of homogeneous materials. It is easy to set up in the laboratory a compact arrangement similar to that of Wente, to which reference has been made (§ 34, § 52), in which electrical devices are employed for producing and detecting sound waves, and to obtain, from a few readings of an alternating current instrument, easily reducible data which are equally as good as those previously obtained by the reverberation method. Here again we have a characteristic application of the newer theory and technology of modern acoustics.

54. *Reverberation in Three Dimensions*

We proceed to apply the principles established in the preceding section, in order to obtain the formulae of Sabine for reverberation in a closed three-dimensional space. Consider first his factor $\boldsymbol{a}$, the total absorbing power of the room. This is not a pure number (that is, a ratio of energy values), but has the dimensions of area; it is defined by the equation

$$\boldsymbol{a} = A_1 S_1 + A_2 S_2 + \ldots + A_n S_n, \tag{277}$$

each term of which is the product of the area S_j of one of the component absorbing surfaces by its absorption coefficient, relative to an equal area of open window space. Thus for six square meters of carpet, of absorption coefficient A_j, we should have a term $6A_j$ for the total absorbing power of this area. It is clear that if we divide $\boldsymbol{a}$ by ΣS_j, we have the "effective" mean absorbing power of the whole boundary surface, which must be identical with the quantity A of § 53. That is to say,

$$A = \frac{\boldsymbol{a}}{S} = \frac{\Sigma A_j S_j}{\Sigma S_j}, \qquad (S = \Sigma S_j), \tag{278}$$

in which it is understood that the summation is carried over all the boundary surface, whatever its configuration, or local absorbing power. This of course implies that all parts of the boundary have equal opportunity to drain energy from the

enclosure, but this is not unreasonable on the static analogy of radiation pressure which we have previously suggested. Experimentally the procedure is amply justified in the accuracy which Sabine usually attained in his practical calculations.

To include the volume V in the analysis we must deal with a factor p, which Sabine called the "mean free path between reflections." In § 53 the mean free path was clearly the length of the tube, (l) just as was stated. By a tedious course of experiment, which need not be described here, Sabine established the relation $p = .62\sqrt[3]{V}$; that is to say, the mean free path between reflections (which is to be regarded as an ideal acoustic dimension, independent of the length, width, or shape of an irregular enclosure) is somewhat less than the cube root of the volume. We can now obtain the general equation for reverberation in a volume V by substituting in (276) the quantity $p = .62\sqrt[3]{V}$ for l, and the ratio $\frac{a}{S}$ for A, thus:

$$\frac{E_2}{E_1} = e^{-\frac{caT}{.62S\sqrt[3]{V}}}. \tag{279}$$

To eliminate the quantity S, we note that for any enclosure of reasonably compact proportions we should have approximately, $S\sqrt[3]{V} = 6V$, since the volume of any elementary cone based on a surface element dS of the boundary of the room is (very nearly) one third of the product of the base by the distance $\frac{\sqrt[3]{V}}{2}\cdot$ Thus we have the fundamental relation

$$\boldsymbol{aT} = \frac{.62(6V)}{c}\log\frac{E_1}{E_2} = \boldsymbol{K}V, \tag{280}$$

the importance of which has already been pointed out. As deduced here, the constant $\boldsymbol{K}$ depends on the determination of the mean free path in terms of the $\sqrt[3]{V}$; Sabine followed the opposite course, first establishing the relation $\boldsymbol{aT} = \boldsymbol{K}V$,

which led him to the relation $p = .62\sqrt[3]{V}$. For the standard ratio $\frac{E_1}{E_2} = 10^6$ (as used by Sabine) we compute, taking $c = 340$ meters per second,

$$K = .15 \text{ sec./meter.} \tag{280a}$$

The meter units of Sabine are retained as best suited to the large spaces dealt with in actual practice. Sabine gives a value $K = .164$ as the best determination[1] of this constant; the discrepancy noted is due to the very approximate character of the calculation we have made, and (probably to some extent) to a relatively poorer determination of the quantity p.[2]

At this point we might well consider the problem of calculating the mean free path p, or what is the same thing, deducing the law of architectual acoustics (eq. 279) *a priori*. Such a calculation was given first by W. S. Franklin (*Phys. Rev.*, XVI, 1903, p. 372); a similar calculation is also given by E. Buckingham[3] and made use of by E. A. Eckhardt in an interesting paper on reverberations.[4]

According to this theory, the number of energy units coming from all directions reaching any bounding surface dS in unit time is $\frac{Ec \cdot dS}{4}$, in which it is assumed that all the trains of incident waves are of energy density E. The amount of energy absorbed in unit time by an area S is then $\frac{AS \cdot Ec}{4} = \frac{aEc}{4}$, in which a is Sabine's area-absorption coefficient. If now $\bar{E}$ is the rate of emission of the source, then the differential equation for the energy density in a closed space of volume V is

$$V\frac{dE}{dt} + \tfrac{1}{4}caE = \bar{E}, \tag{281}$$

[1] *American Architect*, 1900, "Reverberations"; also "Collected Papers," p. 50.

[2] W. C. Sabine, "Papers," p. 40.

[3] *U. S. Bureau of Standards*, Sci. Paper No. 506, May 26, 1925.

[4] *Jour. Fr. Inst.*, 195, 1923, p. 799.

The solution of this equation, for the particular case of decay from an energy density level E_1 is

$$\frac{E}{E_1} = e^{-\frac{ca}{4V}t}; \tag{282}$$

while for the case of growth of energy density to a maximum, the solution is

$$E = \frac{4\bar{E}}{ac}\left(1 - e^{-\frac{ca}{4V}t}\right), \tag{283}$$

since for $t = \infty$, the rate of absorption of energy by the walls of the room must be equal to the generated power.

Equation (282) is identical with (279) if we replace the constant therein ($.62 \times 6 = 3.72$) by 4; or what is the same thing, if we take the mean free path as $p = .67 \sqrt[3]{V}$. Using this theoretical value of p, we have instead of (280*a*), $K = .161$, which is in good agreement with Sabine's experimental value ($K = .165$). This in a way justifies the calculation, which is very rough, from the nature of things. Eckhardt's paper contains a number of graphs illustrating the growth and decay of energy density in rooms of different absorbing power. Before going on we may observe that (282) and (283) make possible a second definition of reverberation time, namely as the *time of decay from the maximum energy density level due to a source of constant power* placed within the room. This conception (due to P. E. Sabine) is one of the newer developments, to which we shall return presently, after a further consideration of the applications which W. C. Sabine made of his ideas.

A characteristic application of W. C. Sabine's theory of Architectural Acoustics was made when he was consulted on the design and interior treatment of Symphony Hall (Boston) in advance of its construction.[1] The plan was virtually to construct a hall whose acoustic quality (reverberation) should be the same as that of the Gewandhaus (Leipzig), but which was to

[1] W. C. Sabine, "Papers," p. 60.

seat a 70 per cent larger audience; when the new hall was completed, its volume was about 60 per cent greater than that of the Gewandhaus. The reverberation time of the Gewandhaus was $T = 2.30$ sec., as calculated by Sabine, due allowance being made for the presence of the audience, in addition to the sum of the absorbing powers of all objects and the boundary surfaces. In collaboration with the architects a design was made for Symphony Hall such that its reverberation time should be 2.31 seconds. This design, together with the prior researches on which it was based, may justly be considered a classic of Applied Acoustics. Sabine makes the interesting point that it would not have done (in the absence of a correct theory) to merely enlarge proportionately the Gewandhaus design; for the reverberation time in that event would have been increased to $T = 3.02$ sec., corresponding to the necessary increase of the volume of the Gewandhaus from 11,200 cu. m. to 25,300 cu. m. to obtain the required seating capacity. This would have differed from the chosen result by an amount that would have been very noticeable. How serious such an error would have been will appear in due course.

To investigate the accuracy of musical taste in judging optimum reverberation time, Sabine carried on a series of tests, cooperating with several competent musicians, leading to the best adjustment of each of five rooms for listening to piano music.[1] When the critics were satisfied with the final adjustments, the reverberation time for each room was measured; a series of values $T = .95$, 1.10, 1.10, 1.09 and 1.16 sec. for the five rooms was obtained. This indicates pretty definitely that the mean reverberation time $T = 1.08$ sec. is characteristic of a room of moderate size acoustically good for this purpose; this result is quite consistent with the results of later work of a similar kind.

P. E. Sabine in a recent paper[2] gives the following rule: "The time of reverberation for an auditorium with its maximum audience as computed by equation [287*a* below] should

[1] W. C. Sabine, *Proc. Am. Acad.*, XLII, 1906; "Collected Papers," p. 71.

[2] P. E. Sabine, "Acoustics in Auditorium Design," *Am. Architect*, June 18, 1924.

lie between one and two seconds. For speech and light music it should fall in the lower half of this range, while for music of the larger sort, it may lie nearer the upper limit." To fully understand this we must discuss his newer idea of reverberation time, to which reference has already been made. If a generator of constant power output $\bar{E}$ is placed in a room of volume V and area-absorption coefficient $\boldsymbol{a}$ the energy density will build up finally to the value

$$E_1 = \frac{4\bar{E}}{ac}. \tag{283a}$$

If decay now takes place from this level, we have, substituting this value of E_1 in (282),

$$E = \frac{4\bar{E}}{ac} e^{-\frac{ca}{4V}t}, \tag{285}$$

whence

$$t = \frac{4V}{ca} \log \frac{4\bar{E}}{caE}. \tag{286}$$

(It is clear that if the ratio of initial to final energy density is

$$\frac{4\bar{E}}{caE} = 10^6,$$

we return at once to W. C. Sabine's first law, namely

$$\boldsymbol{a}\boldsymbol{T} = .16V = \boldsymbol{K}V,$$

as given by (280a), using the accurate value of $\boldsymbol{K}$.) But the purpose of the present calculations is to get away from the concept of a fixed initial energy density and base the definition of reverberation time on a fixed power output at the source. This is because, in practice, we are more likely to deal with sources of constant power than with cases of constant maximum energy density in rooms of different characteristics. And it is

an empirical fact that, with the reverberation time defined on the power basis, there is a better correlation between the reverberation time and the apparent goodness of the room, judged by the artistic test.

Following P. E. Sabine, let us take the power of the source, $\overline{E}$, as 10^{10} cubic meters of sound of threshold density (E) [1], and so obtain a more practical form for (286), which gives the reverberation time T_1 on the new basis as

$$T_1 = \frac{4V}{ca} \log_e \frac{4 \times 10^{10}}{ca}. \tag{287}$$

This is equivalent to [2]

$$aT_1 = .0271V[8.07 - \log_{10} a] \tag{287a}$$

which may be compared to the earlier definition (of W. C. Sabine) given in equations (280, 280*a*); it is equivalent if $8.07 - \log_{10} a = 6$. Broadly speaking, we have in (287*a*) a relation which virtually "corrects" W. C. Sabine's formula $aT = KV$, for unduly large values of a, and gives a new concept of reverberation time more nearly in accordance with the actual conditions under which concert rooms are used.

The following table gives a few comparisons for typical large halls, of T_1 as computed by P. E. Sabine, and T as determined by the older method. These halls are all supposed to be acoustically satisfactory, though not perfect.

[1] "This particular value of $\overline{E}$ has been chosen, since it approximates the acoustic power of the organ pipes used by Professor Sabine in his investigations. Further, a source of sound of this power, will give in empty audience rooms of the usual proportions, with seating capacity of 500 to 1000 persons and with upholstered seats, an intensity of 10^6 times the threshold level. Finally, sound chamber measurements of the power of musical instruments and the human voice indicate that this is a fair approximation to the power of the sounds with which we are dealing in auditorium acoustics." (From an unpublished paper of P. E. Sabine.)

[2] Equation (287*a*) is of course in metric units. If a is given in square feet, and V in cubic feet, we have

$$aT_1 = .0083\, V\,(9.1 - \log_{10} a), \tag{287a}$$

as given by P. E. Sabine, *loc. cit.*

Hall	Volume (cu. m.)	Seats	T_1 (P. E. Sabine)	T	Authority for T
1. Masonic Temple, Madison....	8,580	1500	1.37	1.60	P. E. Sabine
2. Unions Hall, Moscow........	12,500	1600	1.44	1.75	Lifshitz
3. Symphony Hall, Boston......	18,400	2600	1.92	2 31	W. C. Sabine
4. Eastman Theatre, Rochester.	22,400	3340	1.65	2 08	Watson
5. Auditorium Theatre, Chicago	26,200	3640	1.48	1.90	P. E. Sabine

This table contains only a few of the available data, but they are intended to be representative. It will be noted that there is less of a trend toward higher reverberation time with greater volume, for T_1 than for T, even if we exclude No. 1 and No. 5 which represent halls of the theater type, having relatively large seating capacity for a given volume, that is, a larger ratio $a:V$. If data on more halls were given, they would show markedly less variation of T_1 than of T. A general mean of T_1 for large halls of good acoustics has been figured by P. E. Sabine to be 1.47 seconds, and he concludes that this is a very satisfactory standard; it will be noted that this value lies nicely within the range from one to two seconds given in his statement first quoted. And finally we note that the orchestra director Nikish considered the Unions Hall, Moscow, the best hall in Europe; while the Chicago Auditorium is unanimously considered satisfactory for both theater and concert purposes. Considering the diversities of opinion which may be encountered among musical people on the optimum degree of reverberation it seems reasonable, for all practical purposes, to follow the general standard which P. E. Sabine has proposed.

A study of reverberation from the artistic standpoint has been made by S. Lifshitz[1] using the variation in the number of the audience present to vary the absorbing power of the room. For a small room (e.g., for $V < 350$ cu. m.) it was found that the optimum reverberation times for baritone, violin and violoncello music were nearly the same, and centered about the value $T = 1.03$ sec. For a certain orator in a room of volume

[1] Optimum Reverberation for an Auditorium (*Phys. Rev.*, 25, 1925, p. 391). Unfortunately this paper seems to be based on a rather small amount of data.

126 cu. m. the optimum time was $T = 1.06$. These critical observations were apparently made with considerable assurance. For three large halls of volumes 12,500, 13,800 and 17,000 cu. m., the reverberation times were taken to be, respectively, 1.75, 2.00 and 1.55 sec. with all the audience present. The acoustic properties of the first room (the Unions Hall, Moscow) are notably good; the second is too reverberatory; the last is too dead, unless some of the seating capacity is vacant. As a result of experiments of this sort, and certain theoretical considerations [1] Lifshitz proposed a formula for optimum reverberation time as an increasing function of volume. According to this formula the times for the three rooms described should have been, respectively, 1.75, 1.79, and 1.85 sec. On the same basis, Symphony Hall (Boston) would be improved if its reverberation time were lowered to about 1.9 sec. While this may seem somewhat of a refinement over the effect actually striven for in Symphony Hall (i.e., $T = 2.30$), it is fortunate indeed that as a result of acoustic design, the more serious error was avoided of making its reverberation time any greater. The opinion may be hazarded from the preceding discussion that if there is any tendency toward changing our present standards of reverberation, it is likely to be toward lower values.

Recently great advances [2] have been made in recording and reproducing music on the phonograph, with the result that from the standpoint of frequency distortion (that is, resonance, or the undue suppression of high or low frequency sounds), the quality of the reproduction is nearly ideal. In addition, the correct reproduction of energy-level relations (that is, the elimination of distortion due to widely varying amplitudes),

[1] S. Lifshitz, "Architectural Acoustics," Moscow, 1923. In a second paper in the *Phys. Rev.* (27, 1926, p, 618), Lifshitz has clarified his argument and given another empirical equation for optimum reverberation time. He also quotes from Watson's "Acoustics of Buildings" a number of data on American halls. In comparing Lifshitz's conclusions with those of P. E. Sabine, above given, the reader must note that Lifshitz uses T as defined by W. C. Sabine, while P. E. Sabine uses T_1 for his latest data.

[2] Some of these improvements are described by J. P. Maxfield and H. C. Harrison, in a paper on High Quality Recording and Reproducing of Music and Speech, *J.A.I.E.E.*, March, 1926, p. 243.

has been accomplished to a notable degree. These improvements bring forcibly to the attention certain requirements from the standpoint of Architectural Acoustics in the making and use of such records, in order to obtain the illusion of exact reproduction. Clearly the best result is obtained if, when the record is reproduced, the inherent quality in the record and the accompanying effect due to the acoustics of the listening room exactly duplicate the effect that would have been obtained if the original sound had been produced in the regular way in a room of optimum reverberation time. The most straightforward way of insuring this result is to record the original sound with no accompanying reverberation whatsoever, and then to play the record (which is by hypothesis a faithful copy of the original sound) in a room of exactly the acoustic properties demanded for best appreciation of the original sound itself. This is a counsel of perfection, and of course assumes that the user of the phonograph is prepared to take some care in adjusting listening conditions. If this is not practicable, the alternative is for the record maker, at the cost of considerable experimental work, to make less perfect records which are, by a compromise of some kind, better adapted to universal use under average listening conditions. The point we wish to make is that the application of acoustic principles both to recording and to reproduction is now appreciated by the phonograph makers; every effort is made to control recording conditions, and insure a good copy of the original sound; but, granting all this, the best results will not be obtained in final reproduction, without intelligent cooperation on the part of the user of the phonograph, in adjusting the loudness of the sound and the reverberation time of the listening room.

55. *Standing Wave Systems; Focal Properties of an Enclosure; Acoustic Difficulties*

The most unsatisfactory feature of the acoustics of a closed space (equally from the standpoint of theory and of practice) is the inevitable system of standing waves. If the walls have an absorption coefficient of unity, the problem becomes that of an

open space; but reflection is inherent in Architectural Acoustics: in practice the various wall surfaces and wall coverings which are available have absorption coefficients which on the average are far from unity. The absorption coefficients of brick and plaster walls, are for example, 2 or 3 per cent (W. C. Sabine, *loc. cit.* in § 51); a carpeted floor may have a coefficient of 20 per cent; a good grade of acoustic felt, when laid next to the wall, 70 per cent at 1000 cycles (Sabine, "Collected Papers," p. 158); the most extreme value is probably that for the audience, which is given by Sabine as 0.96 in square meter units. Thus the over-all absorption coefficient $\left(A = \frac{a}{S}\right)$ is not likely to be large in any practical case. To take an example, suppose a cubical room of volume 1000 cu. m. has a reverberation time of 1.6 seconds, which would probably fit it for musical purposes; according to equation (280) we should have $aT = .16V$ which gives $a = 100$. Since $S = 600$ for such a room we have $A = \frac{a}{S} = \frac{1}{6}$. Hence, on the average, it would require four reflections to diminish the amplitude of any given wave, as generated, to 49 per cent of its original value. We infer from this that in any enclosure whose "acoustics are good" there will be a sufficient number of multiple reflections to produce standing waves.

Standing wave systems, while easily analyzed in the one-dimensional tube problem which we have frequently studied, are so difficult of analysis in any practical three-dimensional case that they are virtually incalculable. In order to appreciate the complexity of the phenomena encountered in problems of this kind, we shall consider a rather academic example, namely, the steady state wave system corresponding to a small source of sound at one of the foci of an ellipsoidal enclosure having a perfectly reflecting boundary. The problem is several degrees removed from those occurring in practice; rooms do not often have clearly defined sound foci, and if they do, the source of sound is not always located there; rooms never have perfectly reflecting boundaries, nor unbroken walls of regular

shape; consequently such conclusions as we shall draw will be rather inductive. But this example is one of the few which permits of analysis, and is not without a certain interest; there is at least one large audience hall in existence which has nearly this shape.

A section through the foci of the ellipsoid is shown in Fig. 22. To simplify matters the major axis ($2a$) and the distance between foci ($2d$) are each taken equal to an integral number

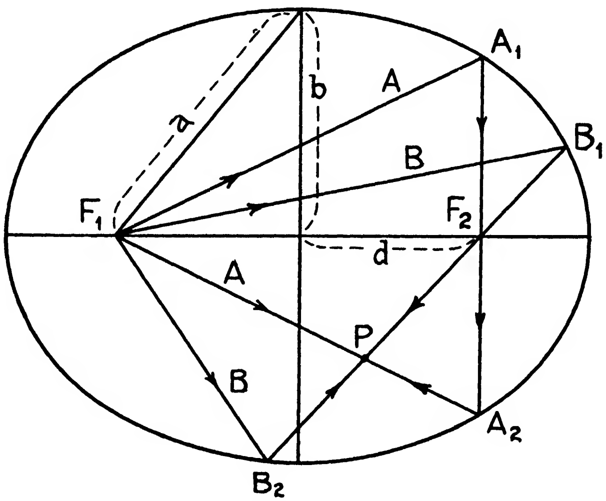

FIG. 22.—GEOMETRY OF THE ELLIPSE.

of wave lengths, e.g., $2a = 8\lambda$, and $2d = 5\lambda$. No great sacrifice in generality is involved in the particular dimensions chosen. The quantities a and d are connected by the relation

$$a^2 - d^2 = b^2,$$

in which $2b$ is the minor axis. From the geometry of the ellipse, the excess pressure at a given point $P = (x, y)$ in the interior will be the sum of the excess pressures inherent in two sets of converging and diverging waves, originating from a source placed at F_1. One set of waves may be said to radiate from the focus F_1 along the lines marked A to P; in this set F_1P is the radius of a divergent wave, and F_2A_2P the radius

of a convergent wave, if (as in the figure) $F_2A_2P > F_1P$. The length $2a$ of the path $F_1A_1F_2$ traversed by the wave through F_2 before entering on the path F_2A_2P need not be taken into account, as this length is 8λ and involves no phase change; the result at P is as if there were two equal sources, in exact phase, located at F_1 and F_2. In other words F_2 is the (positive) image of F_1, to borrow the familiar analogy from electrostatics, or from the theory of mirrors. Similar conditions hold regarding the paths of the other set of waves, which are marked B.

Now in either set of waves meeting at any point P, the radius of curvature of the convergent and divergent components is the same, hence the maximum values of the amplitudes are identical, and only periodic factors (phases) need be considered. Constructive interference (i.e. maxima of excess pressure) in set A will take place whenever $F_2A_2P = F_1P + k\lambda$, k being an integer. The loci of all such points are circles about F_1 as a center, whose radii are $\frac{m\lambda}{2}$, if m is integral. These circles are one set of those represented by the lighter lines in Fig. 23. In addition to this set of circles of maximum excess pressure, there is superimposed another set of such circles, centered at F_2 and similarly derived from the paths B, which also are traversed by the sound waves arriving at P. The crosses which indicate the intersections between the circles of the two sets therefore mark points where the excess pressure is greater than at other points on these circles; for here the pressure maxima in the two systems are additive, since they are in phase. (These points will describe circles normal to the major axis, when the section shown is rotated to generate the ellipsoid.) The regions bounded by circular arcs between the crosses, and containing dashed lines, represent areas of pressure minima, and will develop into rings when the section is rotated about the major axis.

Other interesting properties relate to the general pattern of the standing wave system, and the relative values of pressure and velocity maxima at different points. From one standpoint the crosses are, from their construction, located on con-

focal ellipses, as shown by the dashed lines; or equally, they are located on the corresponding system of confocal hyperbolas, some of which are drawn in. Thus the complete system of regions of maximum pressure may be defined as the circles (and the points on the major axis) which are the intersections of the system of confocal ellipsoids and hyperboloids so spaced that there is a change of phase of a half wave length in passing

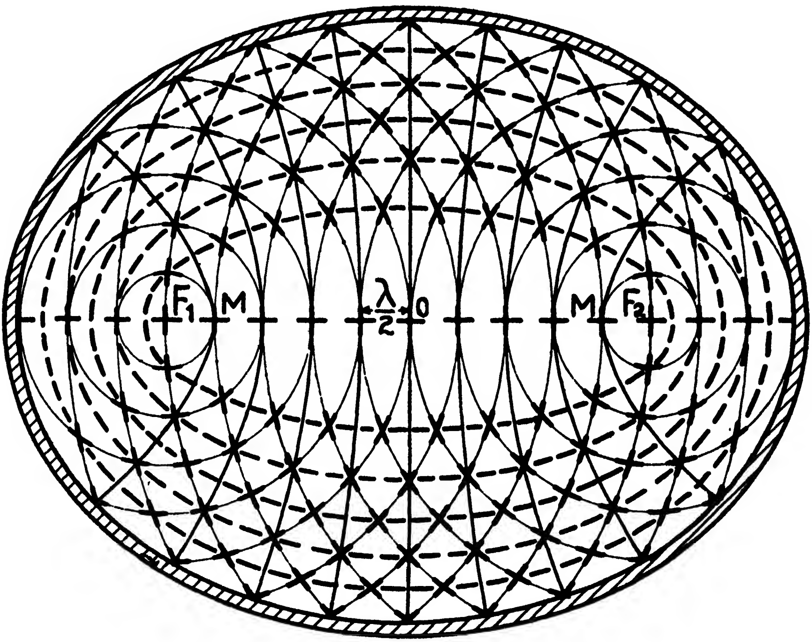

FIG. 23.—SECTION OF THE STANDING WAVE-SYSTEM IN AN ELLIPSOID FOR WHICH $2d=5\lambda$, $2a=8\lambda$.

from one surface to the next. These maxima of pressure are not all of equal intensity or energy density; if we exclude the foci from the discussion, the two crosses marked M in the diagram are the points of most enhanced excess pressure in the whole standing wave system. (This can easily be shown by the reader, and is suggested as a problem at the end of the chapter.) The regions of maximum particle velocity fall in the center of the diamond-shaped and lens-shaped elements de-

fined by the intersecting circles. In those diamond-shaped elements through which the hyperbolas are drawn, the resultant velocity at the center of the element is parallel to the hyperbola, that is, normal to the corresponding ellipsoidal surface; while in the remaining elements the resultant velocities are parallel to the ellipsoidal surfaces; thus there is no normal component of velocity at the boundary. We may observe, however, that in any practical case it is usually sufficient to determine the distribution of pressure maxima, since pressure driven instruments, such as diaphragms or the ear, are most likely to be used for the purpose of detecting the sound. Finally we note that if in any enclosure a point F_2 can be found which is the image of another point F_1 the enclosure is a true whispering gallery [1] that is, all the reflections conspire to make the energy density greatest at F_2 if a source of sound is placed at F_1. It is unnecessary to emphasize further the application of the method of images to the study of sound reflection from curved surfaces which are so regular that they have focal properties.

It is evident that altering the size or the proportions of the ellipsoid will make no essential change in the general distribution of the standing wave system—assuming that the relations between the different parameters are such that a standing wave system is possible at the given frequency. But any change that is made to simulate practical conditions, such as

[1] For a concrete illustration we may refer to the focal properties of the Great Mormon Tabernacle at Salt Lake City. In general, the interior is shaped like a short vertical cylinder of nearly elliptical cross-section, surmounted by a domed roof which is a little too shallow to be half of the corresponding ellipsoid of revolution. In spite of these deviations from an exactly ellipsoidal shape, the enclosure possesses two conjugate foci, and at one of these the speaker's desk is placed. The listening conditions are very unequal in different parts of the enclosure, unless a large audience is present; the best listening point is known to be at the edge of a balcony, in the end of the building opposite the speaker's desk. This is the conjugate focus; here faint sounds (such as the drop of a pin) originating at the speaker's desk can be heard with ease: without reflections from the walls this would be impossible, owing to the great distance between the foci.

A description and photographs of the building are given in W. C. Sabine's paper on Whispering Galleries, previously cited.

for example, making the source of finite size, placing the source out of the focus, or introducing irregularities in the boundary surface will tend to complicate matters so that theory becomes ineffective; it must be emphasized that we have introduced every conceivable geometrical simplification, in order to obtain a theoretical solution. (Even the problem of determining the effect of a uniform lining of absorbing material at the boundary has its difficulties, though it can be solved.) Since *a priori* calculations are impracticable in dealing with most standing wave systems, we must as a rule be reconciled to a series of experimental surveys of the given enclosure, one for each frequency for which the information as to distribution is desired. The experimental procedure is not particularly difficult, and can easily be imagined by the reader; the labor involved is another matter. Astonishing complexities will often be revealed by such an investigation; see for example the result of a survey of a certain room made by W. C. Sabine, shown in Fig. 12, p. 152 of the "Collected Papers."[1] For the benefit of the reader who does not have access to this diagram, it need only be stated that it is made up of contour lines, showing the loci of points of equal energy density, which have every conceivable size, shape and orientation. The only evidence of regularity in the diagram is a certain symmetry about the longitudinal and transverse axes of the room; this effect is natural enough since the source of sound was placed at the center of the room.

In listening to sounds in an enclosure it must be realized that there will be, depending on the frequency, many possible distributions of standing waves. If the sound is at all complex, a number of these patterns are superimposed, and since the maxima and minima for sounds of different frequencies will never exactly coincide, the result will be what we have called *local wave distortion* (§ 50). In listening to a sustained sound no matter how well damped the enclosure may seem to be (by

[1] The Correction of Acoustical Difficulties, *Arch. Quarterly of Harvard Univ.*, March 1912. The diagram also appears as Fig. 8 in the paper Architectural Acoustics, *Jour. Fr. Inst.*, Jan. 1915, which summarizes much of Sabine's work.

a reverberation test), it is easy to get a different impression of the sound by merely shifting the head a short distance. If the sound is a pure tone, the maxima and minima of the standing waves will be in evidence; if the sound is complex, a considerable difference in quality will be noted. If the character of the sound changes as in music or speech, then the standing wave patterns, during the course of the reverberations, are in a state of flux, thus introducing further complications. In listening to concerts in a given hall, it may be found by experience that certain positions are to be avoided, because of excessive local wave distortion, obstructions, or what not; about the only generalization that can be given is that, if the hall is crowded, one should not be too far from the source of sound. We may point out, however, that if the two ears could be located on the surface of one of the reflecting walls of the auditorium, the listener would stand a better chance of correct audition than in the open space of the enclosure, because a reflecting surface is a unique locus of maxima of pressure (velocity nodes) for *all frequencies*. Not only should we expect a more representative frequency distribution of excess pressure at the average point on such a surface, but we should expect less variation in the frequency distribution, from point to point on the wall surface, than from point to point within the enclosure.

The most notorious cases of acoustical difficulty, which experts have been called on to correct, have involved both excessive reverberation and aggravated local wave distortion, focal properties, or echoes. These latter effects of course result in a very unequal distribution of energy density in the enclosure. The first course of treatment is obviously to add absorbing material; to be most effective this should be located near the regions of greatest energy density. If these regions are near regularly curved wall surfaces, a twofold advantage is often gained by placing the absorbing material on such surfaces. If the result is still unsatisfactory after adding all the damping the room will stand (without too great a decrease in reverberation), the regularity of the offending reflecting surfaces should next be modified so as to substitute wave scattering

for regular reflections. This may be done by coffering (as in a domed ceiling) or by the introduction of obstacles (such as large chandeliers) suitably disposed; with the aid of such expedients many originally poor auditoriums have been made serviceable.

For the complete elimination of standing waves, in acoustic experiments, it is possible to employ devices which would not be practicable in an auditorium. The most radical, and probably the most effective device for breaking up the standing wave system, and so insuring a uniform energy density in an enclosure is that used in the W. C. Sabine Laboratory, Riverbank, Geneva, Illinois. This consists of a large steel baffle, or reflecting surface which is slowly and silently rotated about a vertical axis in the room.[1] On the other hand, if it is desired merely to measure the mean energy density produced in a room by a given source under steady state conditions, another expedient may be resorted to; the detector may be placed on a rotating arm, so that the mean reading of the detector will measure the mean energy density around the path traversed.

56. *The Reaction of an Enclosure on a Source of Sound*

It remains to consider the reaction of the enclosure on a sound generating apparatus working in it. The key to this reaction is the impedance of the enclosure, at the point where it is driven; this depends on the state of the standing wave system there. In the mean energy density calculations of § 53 and § 54 it was not necessary to consider the phase of the wave motion, because the rate of working of the source was fixed; but in general to fix the rate of radiation of the source, which is important in physical measurements, the driving point impedance, and its variation from point to point in the enclosure must be determined.

Sabine describes an interesting experiment[2] in which an apparently paradoxical result was obtained. The source of

[1] For a detailed description, see *Sci. Am.*, Sept. 1923, pp. 154–155.

[2] W. C. Sabine, "Collected Papers," Appendix, p. 278.

sound was a diaphragm vibrating with prescribed amplitude at the base of a resonating chamber. The emitted sound was measured when the source was set up in a certain position in the room. The room was then strongly damped by placing felt on the floor, and the emitted sound again measured; it was found to be eight times louder than the original value, for the same amplitude of motion of the diaphragm. The explanation is of course that the introduction of the damping material changed the distribution of the standing wave system, and that whereas, in the first experiment, the diaphragm was located at a point of minimum pressure (or loop of particle velocity), in the second experiment, it was located at a point of maximum excess pressure, and was able to do more work, for a given amplitude of motion, on the surrounding medium.

This explanation can be made more precise if we again apply the familiar theory of the simplest acoustic system: a tube of length l closed at the distant end with a piston of arbitrary impedance. In the discussion following equation (131*a*) (§ 32) we have shown that the condition for maximum power transmission from the driving piston is that in which the tube, by virtue of its length, contributes no reactance to the transfer impedance, and that in this adjustment, the impedance of the distant piston is transferred to the driving point. From (130) the driving point impedance is

$$Z_{00} = \rho c \frac{Z_l \cos \beta l + i \rho c \sin \beta l}{\rho c \cos \beta l + i Z_l \sin \beta l}, \tag{288}$$

and this becomes $Z_{00} = Z_l$ if $\sin \beta l = 0$, that is, if $\cos \beta l = \pm 1$, or $l = \frac{k\lambda}{2}$, k being integral. With this virtual transfer of the distant piston to the driving point, if we let it represent an absorbing boundary, for example, the conditions are those of maximum dissipation at the boundary for a given motion at the source. Again, if the distant piston is to simulate a rigid wall, and we are interested only in increasing the energy density in the tube, using a source of prescribed velocity, the mechanism is that of the Kundt experiment, and the phenomena are as

stated in eq. (135). We now have a maximum energy density in the tube, for the same condition as before, namely $l = \frac{k\lambda}{2}$; that is, the energy density is a maximum if the driving piston is located at a maximum pressure point, since the pressure is a maximum at the distant end.

In certain experiments made in the past, it has been assumed that if the source worked at a prescribed amplitude it was certain to produce, irrespective of its location, a definite energy density in a given space. But if, in the two cases we have considered, the sources were located at pressure minima, instead of pressure maxima, the energy density, and the absorption of power at the boundary, would have been very much reduced; hence the fallacy of ignoring the relative position of the source. Moreover we observe that for any chance disposition of the source, an output more independent of location is likely to result if the source, instead of being driven at constant amplitude, is driven with constant force on the moving element. In this case the amplitude of motion at the source will be restricted partly by its internal impedance, and partly by the driving point impedance of the enclosure, with the result that there is less variation in radiating power as its position is shifted. (These statements can best be illustrated by a numerical example, such as that suggested in problem 49 below.) But we can never eliminate the point to point variation in the impedance against which the source is made to work. and due allowance must be made for this in all acoustical measurements in enclosures.

It is obviously impossible to arrange the members of a choir or orchestra so that each individual source of sound is correctly placed, with respect to the concert hall, for every frequency of tone emitted. On the average, we should expect one compact arrangement of independent sources to be as good as another, assuming that the acoustics of the auditorium are tolerable and that any exaggerated directive effects are eliminated. The conventional arrangement of the orchestra, with the more powerful instruments in the rear, seems to be justified princi-

pally from the standpoint of the conductor, whose judgment and skill in controlling the various parts are aided by having the basic nucleus of the orchestra (the strings) nearest him. In addition, this arrangement gives a good perspective view of the ensemble. But both of these are artistic rather than physical considerations; there is no basic acoustic reason for the accepted arrangement. For one detail of orientation some excuse may be found, on scientific grounds; the horns are supposed to sound better (i.e., to emit tones which sound more mellow) if their bells are not pointed directly at the audience. The sounds they emit are rich in shrill, high-frequency components, so that the source (that is the bell of the horn) is somewhat more directive for these higher harmonics. The effect may become pronounced if a number of horns are played together with the bells in line; the directive effect normal to such a line of sources increases with the length of the line. But in general, in a large room, in which reverberation is taking place, directiveness counts for little: subject always to the consideration that as reverberation becomes diminished by added damping, and conditions approximate more closely to those of open space, any directiveness (or its opposite, divergence) inherent in a particular aggregation of sources will become more and more noticeable. What is best regarding the disposition of an individual instrument can be found by trial; the result is to be judged according to the canons of musical taste. In any event the points mentioned here are secondary, from the standpoint of the listener, to the matter of choosing a good listening point in the auditorium.

. . . .

Here we may well conclude the argument. Throughout, the emphasis has been placed on the physical principles involved; all the problems considered have been chosen with this end in view. It is not to be expected that the reader whose interest lies in Applied Acoustics will be satisfied with the limited applications of the theory which have been made; nor on the other hand, will the mathematically inclined reader be dis-

posed to accept as complete, in every case, the calculations given. To the critical reader of either class, we may observe that the solution of his favorite problem is best left to his own particular devices; and these, of course, must be based on the well-worn and enduring principles of the classical Theory of Sound.

PROBLEMS

41. In Taylor's experiment (p. 109) a layer of sound absorbing material 2 cm. thick reduces the amplitude of the standing waves by 50 per cent (as compared with the amplitudes obtained when the material is removed from the end of the tube) and a layer 4 cm. thick reduces the amplitude by 60 per cent. A further increase in the thickness of the felt produces virtually no effect. What thickness of this material would be required to reduce a transmitted wave to .01 of its original intensity? Certain approximations may be necessary in the solution; note that the reflecting power of the felt is a negative quantity.

42. From Figs. 20 and 21 you can derive the fundamental constants for a certain kind of felt at any frequency within a certain range. A layer of this material 2 cm. thick, when placed on a hard wall, absorbs 50 per cent of the incident energy, at 512 cycles. What is the absorption coefficient of the layer at the same frequency when it is separated from the wall by 5 cm. of air space?

43. A lecture room whose walls and ceiling are of wood (pine) is 12 meters long, 5 meters high, and 8 meters wide. There are four windows on each side, each of area 2 square meters. The room contains 100 chairs for the audience, the floor is covered with a carpet. What is the reverberation time T for the room, with half the audience present, the windows being closed? How is this changed when all the audience is present, and the windows are halfway open, i.e., with all the sash raised? Use the following coefficients in computation:

Absorbing Power Data. (W. C. Sabine)

(In Square Meter Units)

Pine wood surface, per sq. m.,	.06	Chairs, each............	.01
Carpet on Floor, per sq. m.,	.20	Audience, per person.....	.44
Window Glass, per sq. m.,	.03		

44. Two adjoining rooms of volumes V_1, V_2, energy density E_1 E_2, and area-absorption coefficients a_1, a_2, are separated by a partition of area S, and of energy transmission coefficient k per unit area. A source whose energy rate is E is working in V_1. What are the maximum steady state values of E_1 and E_2? Show that, under certain conditions, the constant k can be determined from the relation

$$\log\left(\frac{a_2}{kS}\right) = \frac{ca_1}{4V_1}(T_1 - T_2),$$

if T_1, T_2 are the respective reverberation times of V_1, V_2. If there is no reverberation in V_2, what is the intensity of the sound on emerging from the partition? (A. H. Davis, *Phil. Mag.*, 50, July, 1925, p. 75.)

45. Derive an algebraic expression for the reverberation time of a spherical enclosure lined with absorbing material, assuming that the original distribution of energy density is that of the steady state produced by a constant small source placed at the center.

46. A certain auditorium contains 20,000 cu. m., with seating capacity of 3,000. With all the seats occupied, such a hall might possibly have a total absorption coefficient (W. C. Sabine) of 2×10^3 sq. m. units. Is the room acoustically good?

A quarter note is sounded in this auditorium by an instrument working at the rate of 10^{10} threshold units of energy per second, and the note is maintained for 0.5 second by the player. What is the average energy density at the end of the note, in terms of the threshold level? What is the time of decay (T_1) to the threshold level?

47. In the ellipsoid of § 55 show that the energy density is greatest at the points of maximum pressure nearest the foci.

48. In Wente's experiment (p. 109) find the relation between the maximum and minimum driving-point impedances of the tube, and the reflection and absorption coefficients of the layer at the end of the tube.

49. A piston whose impedance is $a \cdot R$ (a pure resistance) is used to drive one end of a tube of air, of unit section, R being the radiation resistance of air. The other end of the tube is closed with a layer of absorbing material whose impedance is also $a \cdot R$. Two adjustments of tube length are made: (1) $l = \frac{k\lambda}{2}$, and (2), $l = \frac{(k + \frac{1}{2})\lambda}{2}$. Show

that, if the piston is driven at a prescribed velocity in both adjustments, the relative rates of radiation are as a^2: 1. Show also that if the piston is driven with a prescribed force in both cases, the relative rates of radiation are as

$$\frac{a^2}{4} : \frac{a^4}{(a^2 + 1)^2}.$$

NOTE.—In all practical cases a is likely to be greater than unity.

50. What is the driving point impedance of a spherical enclosure, if a small spherical source is placed at the center?

APPENDIX A

RESISTANCE COEFFICIENTS FOR CYLINDRICAL CONDUITS

> " *The viscosity of a substance is measured by the tangential force on unit area of either of two horizontal planes of indefinite extent at unit distance apart, one of which is fixed, while the other moves with unit velocity, the space between being filled with the viscous substance.*"
>
> —MAXWELL'S DEFINITION.

A fuller consideration may be worth while of the phenomena of resistance to fluid motion in tubes of circular section, which effects have been rather casually treated in the text. Two cases must be distinguished, according as the section of the tube is effectively small or of moderate size, the classification depending not only on the relation of the constants of the fluid to the actual width of the tube, but also on the frequency at which the fluid is driven. As will appear, the discriminant between the two cases is the quantity

$$kr = r\sqrt{-\frac{i\rho\omega}{\mu}},$$

in which r is the radius of the tube. If $|kr|$ is not greater than unity, the tube is effectively a " narrow " tube; the reaction due to the inertia of the fluid is much less than the frictional resistance, and ratio of the driving force to the mean velocity over the section is Poiseuille's Coefficient $\left(R = 8\frac{\mu}{r^2}\right)$, neglecting the inertia component. In this case there is lamellar motion throughout the section, the velocity varying from zero at the wall of the tube to a maximum at the center: this condition is

illustrated roughly in Fig. 24*a*. Theoretically the situation is analogous to that in a vibrating system when both mass and damping are adding to the system; but the damping or choking effect in the tube is so great that compressional waves are transmitted only with great difficulty, and the resemblance is to an aperiodic system.

In § 32 the problem of wave transmission was treated on the general assumption that some resistance coefficient R_1 was applicable which was small compared with $\rho\omega$. For the steady

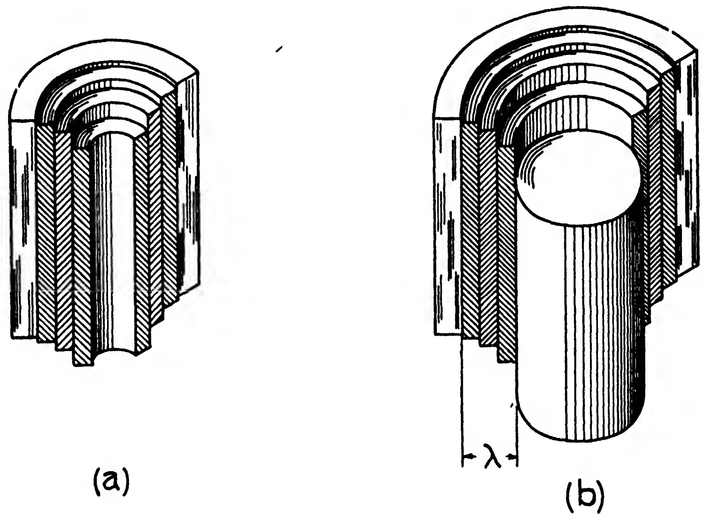

FIG. 24.—LAMELLAR MOTION OF FLUID IN SMALL AND MODERATELY LARGE TUBES.

flow of liquids in pipes, it is doubtless sound enough to apply Poiseuille's Coefficient; but for pipes of such sizes as are likely to be used for sound transmission, as for example in the Constantinesco scheme, it would seem that the inertia of the fluid plays such a dominant role that Poiseuille's law no longer applies.

It must be admitted that here, as in certain other hydrodynamical problems, the reconciliation of theory and practice is by no means complete and it may well be that empirically determined coefficients are necessary in order to deal accurately

with this situation. It would be well, for example, if the upper frequency limit for which Poiseuille's coefficient is valid could be established experimentally; the ease of computation according to a simple formula of this sort would be of advantage in many cases. Unfortunately reliable data of this kind are not available.

It happens that in most problems involving sound transmission in tubes, the factor $|kr|$ is large as compared with unity; in other words the tube is effectively "large" and does not choke or damp the oscillations to a very great degree. The situation is then analogous to a mildly-damped vibrating system, the effect of the damping being to decrease the velocity of propagation by a small quantity of the second order. This defect in velocity was computed by Helmholtz, whose solution of the problem we shall follow. There have been numerous experimental tests, which we shall refer to later, of the Helmholtz formula; unfortunately these are not concordant, nor do they fully support the simple theory. The purpose of the following discussion is therefore limited to placing before the reader the best available idea of the mechanism of the losses due to viscosity, and the essential difference between the effects in wide and in narrow tubes.

The motion in the case of the wider tube is roughly as shown in Fig. 24*b*. Owing to the oscillation of the fluid along the axis of the tube, viscosity waves are diffused radially from within the fluid (where the velocity is greatest) toward the walls of the tube, where the velocity is *nil*. It is a peculiar property of diffusion waves that they are virtually extinguished after traversing a distance of one wave length. One result of this is that the effect of viscosity, for sound waves in tubes wider than (say) double this wave length, is confined to a shell of thickness approximately one wave length next to the wall of the tube. There remains a region in the center of the tube in which the dragging effect is practically absent, and in which the axial velocity at any point does not vary greatly with its distance from the center. The conception is then of a cylindrical core of air, oscillating as a unit in the center of the tube, and

impeded in its motion by the reactions which take place in a thin layer of fluid between its cylindrical boundary and the inner wall of the tube. These reactions involve both added inertia and resistance as can be seen from the following considerations.

Let an infinite plane wall oscillate in its own plane in contact with the fluid, with the result that viscosity waves are diffused in the X direction, normal to the oscillating plane. The force due to the inertia of an element of fluid of unit area and thickness dx is $\rho\ddot{\xi}dx$. The net force on the element due to shearing stress is (by definition of μ) $-\dfrac{\partial}{\partial x}\left(-\mu\dfrac{\partial\dot{\xi}}{\partial x}\right)dx$, the negative velocity gradient being used because of the falling away in velocity with increasing x. The equation of motion is therefore

$$\frac{\partial\dot{\xi}}{\partial t} = \frac{\mu}{\rho}\frac{\partial^2\dot{\xi}}{\partial x^2}, \tag{A}$$

and if $\dot{\xi}(x) = \dot{\xi}_0 e^{i(\omega t - k'x)}$ we have

$$k' = \pm\,\beta(1 - i), \qquad \beta = \sqrt{\frac{\rho\omega}{2\mu}}; \qquad \text{since } \sqrt{-i} = \frac{1 - i}{\sqrt{2}}.$$

The solution is therefore, for waves in the positive direction,

$$\dot{\xi}(x) = \dot{\xi}_0 e^{-\beta x} e^{i(\omega t - \beta x)} \tag{B}$$

and $\lambda = \dfrac{2\pi}{\beta} = 2\pi\sqrt{\dfrac{2\mu}{\rho\omega}}$. This is the distance to which the transverse vibrations are of any consequence: for $e^{-\beta\lambda} = e^{-2\pi}$, a very small quantity. The reaction $R\dot{\xi}$ on the driving wall is, per unit area, the product of the viscosity and the velocity gradient, that is, dropping the exponential factor,

$$R\dot{\xi}_0 = -\mu\left|\frac{\partial\dot{\xi}}{\partial x}\right|_{x=0} = \mu\beta(1 + i)\dot{\xi}_0, \tag{C}$$

which gives a resistance coefficient $R = \mu\beta\,(1 + i)$. We note that the imaginary part of this expression is in phase with the

acceleration of the driving surface, and is therefore of the nature of a mass reactance; it is evident, in applying this to the tube problem that there will be a slowing down of the propagation velocity of the sound waves in the tube, because of the mass reactance on the oscillating core of the medium in the tube, the effect of which is virtually to increase its density. This corresponds to the lowering of the natural frequency of a vibrating system by loading it with added mass, and is a larger effect than that due to the added damping.

The total force on unit length of the surface of the core of moving gas in the tube is $2\pi r.R$; so that, per unit sectional area of the tube the resistance coefficient is

$$R_1 = \frac{2\pi r}{\pi r^2}\cdot R = \frac{2\mu\beta}{r}(1 + i), \tag{D}$$

In applying (C), obtained for plane waves, to the surface of a cylinder, and so deriving equation (D), the curvature of the cylindrical surface has been neglected. This is legitimate, for the thickness λ of the region between the oscillating core, and the wall of the tube is relatively small as compared to the radius of the tube in which these effects are supposed to take place. (For example $\lambda = .6$ mm. for a frequency of 500 in air.)

The results obtained above are confirmed by a more general treatment, which takes into account the cylindrical structure of the tube, and the whole range of driving frequencies; and in addition an insight is gained into the meaning of the criterion first given, which depends on the magnitude $|kr|$ for differentiating between the two types of resistance phenomena.

We proceed as in § 10, except that the circular section (πr_0^2) of fluid is substituted for the circular membrane, the axial driving force now being $\Psi\cdot dx$ per unit area; Ψ is of course the negative pressure gradient parallel to the axis of the tube. The total driving force on an annular ring of fluid of volume $2\pi r dr\cdot dx$ is $\Psi dx\cdot 2\pi r dr$; this is opposed by a reactance $i\omega\rho\cdot 2\pi r dr\cdot dx$ due to inertia. The opposing force due to friction on the *inner* surface of the ring is $-\ 2\pi r dx\cdot\mu\frac{\partial\xi}{\partial r}$, using the

negative velocity gradient as before, because of decrease of $\dot{\xi}$ with increasing r. The net force on the annulus due to friction is therefore

$$\frac{\partial}{\partial r}\left(-\ 2\pi r\ dx\ \mu\frac{\partial\dot{\xi}}{\partial r}\right)dr,$$

and hence the equation of motion

$$\left[i\omega\rho - \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)\right]\dot{\xi} = \Psi, \tag{E}$$

in which only $\dot{\xi}$ is a function of r. This may be written

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + k^2\right]\dot{\xi} = -\frac{\Psi}{\mu} \qquad \left(k^2 = -\frac{i\rho\omega}{\mu}\right), \tag{E'}$$

the solution being

$$\dot{\xi} = -\frac{\Psi}{\mu k^2} + AJ_0(kr), \tag{F}$$

for finite velocity when $r = 0$. [Cf. eq. (32.)] Exactly as before (for the membrane) the velocity must vanish at the boundary, $r = r_0$; determining A we have

$$\dot{\xi}(r) = -\frac{\Psi}{\mu k^2}\left[1 - \frac{J_0(kr)}{J_0(kr_0)}\right]. \tag{F'}$$

Integrating $\dot{\xi}(r)$ over the section, we have for the mean velocity

$$\bar{\dot{\xi}} = \frac{2}{r_0^2}\int_0^{r_0}\dot{\xi}\,r\,dr = -\frac{\Psi}{\mu k^2}\left[1 - \frac{2}{k^2 r_0^2 J_0(kr_0)}\int_0^{r_0}J_0(kr)\cdot kr\cdot k\,dr\right],$$

that is

$$\bar{\dot{\xi}} = -\frac{\Psi}{\mu k^2}\left[1 - \frac{2}{kr_0}\frac{J_1(kr_0)}{J_0(kr_0)}\right], \tag{G}$$

according to the method of § 12, eq. (47). The equation is now in the form $\dot{\xi} = \frac{\Psi}{R}$ and it only remains to discuss the values taken by R as a function of frequency.

If $|kr|$ is not greater than unity we may take

$$J_0(kr_0) = J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}, \qquad \text{(cf. 25)}$$

and

$$J_1(kr_0) = J_1(x) = \frac{x}{2}\left(1 - \frac{x^2}{8} + \frac{x^4}{192}\right),$$

using only 3 terms of these well known series expansions. We then have,

$$\left[1 - \frac{2}{x}\frac{J_1(x)}{J_0(x)}\right] = \left[\frac{1 - \frac{x^2}{4} + \frac{x^4}{64} - 1 + \frac{x^2}{8} - \frac{x^4}{192}}{1 - \frac{x^2}{4}}\right],$$

$$= -\frac{x^2}{8}\frac{\left(1 - \frac{x^2}{12}\right)}{\left(1 - \frac{x^2}{4}\right)} = -\frac{x^2}{8}\frac{1}{\left(1 - \frac{x^2}{6}\right)}.$$

Thus (G) becomes

$$\dot{\xi} = \frac{\Psi r_0^2}{8\mu}\frac{1}{\left(1 - \frac{k^2 r_0^2}{6}\right)} = \frac{\Psi}{R}, \qquad (H)$$

and

$$R = \frac{8\mu}{r_0^2} + \frac{4}{3}i\omega\rho, \qquad \text{since } k^2 = -\frac{i\omega\rho}{\mu}. \qquad (I)$$

The pure resistance term in this equation is *Poiseuille's Coefficient*, $R_1 = \frac{8\mu}{r_0^2}$; the reactance $\frac{4}{3}i\omega\rho$ represents a $\frac{1}{3}$ increase in effective mass or density due to the diffusion effects in the narrow tube, as compared with the reactance $i\rho\omega$ in the unlimited medium. The inertia component is of little importance, i.e., if $|kr_0| = 1$, the inertia reactance is only $\frac{1}{6}$ of the resistance factor: the whole behavior of the tube as a wave-transmitting system is profoundly modified, and in fact approximates to an

aperiodic system, as has been previously stated. The equation of wave motion, in terms of mean velocity, is

$$R\dot{\xi} = \left(\frac{4}{3}i\rho\omega + \frac{8\mu}{r^2}\right)\dot{\xi} = \kappa'\frac{\partial^2\xi}{\partial x^2}, \tag{K}$$

which becomes, if we neglect the reactance

$$\frac{i\omega R_1}{\rho}\xi = c^2\frac{\partial^2\xi}{\partial x^2}; \qquad c^2 = \frac{\kappa'}{\rho}, \qquad R_1 = \frac{8\mu}{r_0^2}. \tag{K'}$$

This is in the same form as eq. (A), and solving in the same manner we have

$$\xi = \xi_0 e^{-\beta x} e^{i(\omega t - \beta x)} \tag{B'}$$

in which

$$\beta = \frac{1}{c}\sqrt{\frac{\omega R_1}{2\rho}} = \frac{\omega}{c}\sqrt{\frac{4\mu}{\rho\omega r_0^2}} = \frac{\omega}{c}\frac{2}{|kr_0|^2}.$$

The phase velocity is now

$$c' = c\frac{|kr_0|^2}{2}, \tag{L}$$

in which c is as usual, the unmodified velocity of sound. It is evident at once that the phase velocity is very greatly reduced, this effect being inevitable as the sequel to the high damping and attenuation in the narrow tube.

To evaluate R for values of $|kr_0|$ in the range $1 < |kr_0| < 10$ it is necessary to make use of the relation

$$J_0(x\sqrt{-i}) = \text{ber } x + i \text{ bei } x,$$

in much the same way as was done in § 11 to solve the air damping problem. We shall not enter into these rather prolix calculations because in practice the values of R in this frequency range are not of much concern. If r_0 is 1 cm. and $\omega > 20$, $|kr_0| > 12$ (for example) in air; thus for all acoustic frequencies we can rigorously apply certain simple formulae to evaluate the Bessel's functions, whose arguments are then complex variables of large absolute value.

From (G) we have

$$\frac{1}{R} = -\frac{1}{\mu k^2}\left[1 - \frac{2}{kr_o}\frac{J_1(kr_o)}{J_o(kr_o)}\right], \tag{G'}$$

to which we apply the relation (cf. Jahnke and Emde, p. 101),

$$\frac{J_1(x\sqrt{-i})}{J_o(x\sqrt{-i})} = -i, \qquad x\sqrt{-i} = r\sqrt{\frac{-i\rho\omega}{\mu}}, \tag{M}$$

which is very nearly true for $x > 10$. We then have

$$\frac{1}{R} = -\frac{1}{\mu k^2}\left[1 + \frac{2i}{kr_o}\right],$$

or

$$R = -\mu k^2\left[1 - \frac{2i}{kr_o}\right] = i\rho\omega\left[1 + \frac{2\sqrt{-i}}{r_o}\sqrt{\frac{\mu}{\rho\omega}}\right].$$

This reduces to

$$R = i\rho\omega + \frac{2\mu\beta}{r}(1 + i), \qquad \beta = \sqrt{\frac{\rho\omega}{2\mu}}, \tag{N}$$

the second term of which confirms the result previously obtained for the large tube [cf. eq. (D)]. The inertia reactance $i\rho\omega$ is the normal inertia effect present in any case, if there are no viscosity effects. The principal interest in this solution of the resistance problem lies in the additional reactance $\frac{2i\mu\beta}{r}$ due to the combined viscosity and inertia effects in the tube of moderate width; this gives rise to a second order effect of diminished propagation velocity.

The equation of wave motion is now, since $i\dot{\xi} = \frac{\ddot{\xi}}{\omega}$,

$$\left(\rho + \frac{2\mu\beta}{\omega r}\right)\frac{\partial^2 \xi}{\partial t^2} + \frac{2\mu\beta}{r}\frac{\partial \xi}{\partial t} = \kappa'\frac{\partial^2 \xi}{\partial x^2}, \tag{O}$$

or, say,

$$\rho'\frac{\partial^2 \xi}{\partial t^2} + R'_1\frac{\partial \xi}{\partial t} = \kappa'\frac{\partial^2 \xi}{\partial x^2}. \tag{O'}$$

Solving this equation exactly as before (§ 32, p. 96–97), we have the following relations:

$$\left.\begin{aligned} \xi &= Ae^{-\alpha' x}e^{i(\omega t - \beta' x)}, \\ \alpha' &= \frac{R'_1}{2\rho' c'}, \quad \beta' = \frac{\omega}{c'}, \end{aligned}\right\} \qquad (P)$$

in which it will be noted that

$$c' = \sqrt{\frac{\kappa'}{\rho'}} = \frac{\sqrt{\kappa'}}{\sqrt{\rho\left(1 + \frac{2\mu\beta}{\omega\rho r}\right)}} = c\left(1 - \frac{\mu\beta}{\omega\rho r}\right), \qquad (Q)$$

approximately. Remembering that $\beta = \sqrt{\frac{\rho\omega}{2\mu}}$ [eq. (B)] we may summarize the results in terms of the simpler constants thus:

$$\left.\begin{aligned} c' &= c\left(1 - \frac{1}{r}\sqrt{\frac{\mu}{2\rho\omega}}\right), \qquad \rho' = \rho\left(1 + \frac{2}{r}\sqrt{\frac{\mu}{2\rho\omega}}\right) \\ \rho' c' &= \rho c\left(1 + \frac{1}{r}\sqrt{\frac{\mu}{2\rho\omega}}\right), \quad R_1' = \frac{2\mu\beta}{r} = \frac{1}{r}\sqrt{2\rho\mu\omega}, \\ \alpha' &= \frac{R_1'}{2\rho' c'} = \frac{1}{rc}\sqrt{\frac{\omega\mu}{2\rho}}, \qquad \text{approximately.} \end{aligned}\right\} \qquad (R)$$

As an example of the magnitude of the attenuation effect, according to the Helmholtz formula, let the radius of the tube be 5 mm., and $\omega = 6000$, in air. The change in velocity is $c - c' = .007c$; and $\alpha' = 1.3 \times 10^{-3}$. It is evident therefore that in many problems these effects are so small that we are justified in neglecting them.

These calculations of course ignore the loss due to heat conduction to the wall of the tube, which is a more difficult effect to deal with, though the calculations are not dissimilar. The diffusivity of a gas is a coefficient of the same order of magnitude as the kinematic viscosity, hence the effect of losses due to heat conduction will be equivalent to a sensible increase in the kinematic viscosity, in the general case. In the special case of a

narrow metal tube, if there were no viscosity to contend with, the volume changes in the gas would take place isothermally, just as we have pointed out in § 11 when calculating the pressure changes in a thin film of air bounded by metal surfaces. In such a case the velocity of propagation would become in the limit, that of the Newtonian formula, $c^2 = \frac{p_0}{\rho}$. But we have already seen [eq. (L)] that in an arrow tube, on account of viscosity alone, the phase velocity is very greatly reduced. In this particular case the effect of viscosity predominates.

According to the extended treatment of Kirchoff's contribution to the theory (Rayleigh, II, § 348), which takes account of the effects of heat conduction, we should add to the factor $\nu = \frac{\mu}{\rho}$ a factor (σ, say) which represents the "diffusivity" or "temperature conductivity" of the gas. If we write $(\gamma')^2 = \nu + \sigma$, then we have for the modified velocity of sound in the tube of moderate width

$$c' = c\left(1 - \frac{\gamma'}{2r\sqrt{\pi f}}\right), \tag{R'}$$

on which basis most investigators have reduced their results. For air at normal temperature and pressure we should have $\nu = .13$, $\sigma = .17$, and hence $\gamma' = .54$, in c.g.s. units.

We may finally discuss some of the experimental tests that have been made of the Helmholtz–Kirchoff formula; the older work is well reviewed in Barton's "Textbook of Sound," Chap. X (§§ 516–529). Following the work of Kundt (who first observed the defect in velocity, in a tube), Schneebeli (1869) and A. Seebeck (1870) experimented with tubes of various diameters, using sound waves of somewhat limited frequency range. From these experiments it appeared that the defect in velocity varied as $\frac{1}{r}$, in accordance with the formula, but it varied with frequency as $\frac{1}{\omega^{\frac{1}{3}}}$, not as $\frac{1}{\sqrt{\omega}}$. Kaiser,

later, was enabled to reconcile his tests with the formula provided μ were given a large value: a proposition not inconsistent with increasing μ to take account of losses due to both heat conduction and viscosity. Blaikley (1884), after very careful tests at a single frequency (256), using *smooth brass* tubes of diameters from 1 to 9 cm., came to the conclusion that the Helmholtz formula was correct as to variation of the effect in accordance with the inverse diameter of the tube; the values obtained also favored the view that the effect was inversely proportional to $\sqrt{\omega}$. But in 1903 J. Müller concluded that the Helmholtz formula was not valid, and that the speed of sound waves in a tube depended among other things on the material of the tube. (This is doubtless true, to a certain extent, as we have pointed out in § 32; but we should surely expect the yielding of a thick metal wall to be a very small factor, for sound waves in a gas within.) F. A. Schulze (1904) experimented with very *narrow* tubes (ca. 1 mm. diameter), but we should not expect these results to bear on the Helmholtz formula, for reasons we have given.

In the work of E. H. Stevens, (*Ann. d. Phys.*, 7, 1902, p. 285) since he was interested in the velocity of sound *per se*, the correction factor γ' was eliminated, by comparing measurements made with tubes of various diameters—a common practice in work of this kind. He gives a discussion of the previous experimental studies of the Helmholtz-Kirchoff effect, and obtained from his own measurements a value for γ' (air) somewhat larger than that predicted by theory. But the important point is that for tubes from 2 to 4 cm. in diameter, he found γ' constant, and hence adopted the Helmholtz-Kirchoff formula. E. Grüneisen and E. Merkel (*Ann. d. Phys.*, 66, 1921, p. 344) in a similar study of the velocity of sound, obtained for γ' the value 0.49 c.g.s. (air) as compared with the theoretical value of 0.54 c.g.s.

From all these experiments it seems reasonable to conclude that in some of the earlier work inconclusive results were obtained through failure to control conditions so that a straightforward test of the Helmholtz-Kirchoff Formula was possible. There

seems to be an opportunity for an extended series of measurements on the transmission of sound waves of a wide range [1] of frequency, in good smooth tubes. And until comprehensive transmission data of this sort are available, it seems well to abide by the outline of theory we have considered.

[1] As bearing on the frequency range to be covered, we note an experimental study of the transmission of very low frequency air waves in a long rubber tube, made by Simmons and Johansen, *Phil. Mag.*, 50, 1925, p. 553. The results show that the attenuation in the tube is greater than that given by the theory; this may be due to the yielding of the wall (§ 32, note). The wave velocity for the low frequencies used (30 to 60 per. per min.) was somewhat less than the Newtonian value (that is, less than 278 meters per second). The observations quoted relate to a pipe 9.5 mm. in diameter; the authors found that when the diameter was reduced, a very considerable reduction in wave velocity took place.

APPENDIX B

Recent Developments in Applied Acoustics

To the reader interested in acoustic research or technology the following notes[1] may supply such references as are needed to supplement those given in the text, in order to get quickly into touch with current practice. No specific references are added to those already given in the discussion of (1) Acoustic Filters (§ 27); (2) Tubes and Pipes (§ 33, § 43; Appendix A); (3) High Frequency Underwater Sound Devices (§ 41); (4) Horns (§ 45, § 46, § 47) and (5) the Acoustic Radiometer (§ 48). With the exception of tubes and pipes, on which a great deal of older acoustic work has been done, there is little further dependable information to offer on these devices. Some supplementary data on the condenser transmitter (cf. the air-damped vibrating system, § 11) and on resonators (cf. § 24) may be of interest, in addition to the necessary introductory notes on other experimental apparatus.

For a useful general bibliography of some comparatively recent work, reference is first made to a *Bulletin of the National Research Council*[2] which was prepared by a committee of representative American Physicists in 1922. For economy of space, most of the references to be given will concern work published since that time.

Piezo-Electric Resonators

A brief discussion of a typical experiment will serve to illustrate the contrast between the older technique and the

[1] These have been prepared with the able assistance of Miss H. M. Craig, Research Librarian of the Technical Library, Bell Telephone Laboratories. In addition to supplying many of the references, she has taken every possible care to verify the entire list.

[2] Vol. 4, Part 5, Nov., 1922, " Certain Problems in Acoustics."

new; we select for the purpose Kundt's experiment, the theory of which has been given in § 32. For an account of the original experiment we may turn to Barton's[1] "Text Book of Sound," (London, 1914), (p. 530), which contains a very comprehensive Chapter (X) on Acoustic Determinations. It is of interest to note that in 1868 Kundt, with his hand-operated rod and tube, measured the relative velocities of sound in various fluids, and determined the effect on the velocity of narrowing the tube, which (as we have seen in Appendix A) decreases the velocity more or less according to the Helmholtz formula. According to most recent practice the resonant metal rod is equipped at the pressure maximum (the node, or point of support, half way along the length) with piezo-electric quartz crystals, and the electro-mechanical system thus constituted is placed as a condenser in an electrical oscillator circuit. With this apparatus W. G. Cady (*Phys. Rev.*, 21, 1923, p. 371; *ibid.*, 23, 1924, p. 558), using electrically driven rods whose natural frequencies are very accurately known, has provided a convenient and precise method for determining the velocity of sound in tubes.

The theory of longitudinal vibrations of a viscous rod is given by Cady in *Phys. Rev.*, 19, 1922, p. 1, and the theory of the vibrating piezo-electric crystal follows in *Proc. I.R.E.*, 10, 1922, p. 83; the determination of the equivalent electric network of the system is further treated by K. S. Van Dyke, *Phys. Rev.*, 25, 1925, p. 895. There is also a paper by M. v. Laue (*Zeit. f. Phys.*, 34, 1925, p. 347) on piezo-electric oscillations in quartz rods. The piezo-electric resonator is inherently stable and permanent, and has been developed in various forms by Cady as a frequency standard, with particular application to radio frequency measurements; see *Proc. I.R.E.*, 12, 1924, p.

[1] The late E. H. Barton (1859–1925) a Fellow of the Royal Society, was a consistent contributor to the literature of Acoustics. In collaboration with his students he wrote many papers dealing with the performances of musical instruments, a subject which he was specially qualified to discuss. These papers have appeared (interspersed with his papers on resonance, and on coupled systems) for some years in the *Philosophical Magazine;* the latest one will be found in vol. 50 (1925) p. 957, and deals characteristically with the tones of the trumpet and the cornet. A biographical notice of Prof. Barton appeared in *Nature*, Nov. 7, 1925, p. 685.

805; also *Jour. Opt. Soc. Am.*, 10, 1925, p. 475; also F. E. Nancarrow, *P.O.E.E.J.*, 18, 1925, p. 168; also G. W. Pierce, *Proc. Am. Acad.*, 59, No. 4, 1923, p. 81. The use of electrically controlled tuning forks for standardizing frequency is described by Horton, Ricker and Marrison, *Trans. A.I.E.E.*, 42, 1923, p. 730.

Piezo-electric crystal oscillators have been applied by G. W. Pierce (*Proc. Am. Acad.*, 60, No. 5, 1925, p. 271) to the precision measurement of the velocity of sound over a range of high frequencies (4×10^4 to 10^6 cycles) in air and in carbon dioxide. The method depends on the reaction, at the driving point, of a standing wave system produced in the medium between the driving crystal and a nearby reflecting wall. (The action of the piezo-electric system is similar to that of other electro mechanical devices in that the velocity of its moving member is translated into an electrical "motional" impedance, which, added to the inherent electrical impedance of the apparatus, determines the net electrical impedance between terminals. Thus from one standpoint, in an experiment of this kind, we have a problem in impedances quite similar to those we have met in the preceding text; or if we prefer, we may by analogy call Pierce's apparatus an Acoustic Interferometer.) Among the experimental results obtained by Pierce are, an apparent frequency variation in the velocity of sound in both gases, and a large sound absorption effect by carbon dioxide at high frequencies. Low-frequency determinations based on resonance in a 4 cm. tube were also made, using a simple telephone receiver as a source. The factor for the defect in velocity due to the tube was eliminated in reducing the observations. The paper also gives references to a number of modern determinations of the velocity of sound in air.

These references, with those below on the telephone receiver, are cited to emphasize the important field of work in which acoustics and alternating currents have become very closely associated. This development has not only greatly increased the resources of the acoustic laboratory, but has also resulted in better working ideas in the theory of vibrating systems generally.

The Telephone Receiver

This instrument, invented by Alexander Graham Bell in 1875, is not only one of the oldest, but has become one of the most indispensable devices of the laboratory. References have already been given (§ 7) to the work of A. E. Kennelly and his associates on the motion of the diaphragm, and other matters which are treated in "Electrical Vibration Instruments." A good introductory paper on the telephone is that of L. V. King, *Jour. Fr. Inst.*, 187, 1919, p. 611. The most general treatment of the receiver is that of R. L. Wegel (*J. A.I.E.E* 40, 1921, p. 791). Two papers by Hahnemann and Hecht are referred to in the footnote (§ 7) p. 19. Their work has been continued in two papers on the Receiver in *Ann. d. Phys.*, 60, 1919 p. 454; 63, 1920, p. 57; and 64, 1921, p. 671. Two additional articles by these authors, translated in *Engineering*, 106, 1918, p. 756 and 107, 1919, p. 224, deal with sound radiation and sound generators, in a different style[1] from that adopted in the present text. A recent paper by Mallet and Dutton, *Jour. I.E.E.*, 63, 1925, p. 502 (discussion later, *ibid.*, p. 715), describes some interesting acoustic experiments with receivers, and gives references to earlier work by these writers.

The Standard Phone, and the Phonometer of A. G. Webster, used for sound-intensity measurements, are described in *Proc. Nat. Acad. Sci.*, 5, 1919, p. 173 and 275.

The best paper on the behavior of a diaphragm immersed in a sound field is that of D. A. Goldhammer (*Ann. d. Phys.*, 33, 1910, p. 192). A paper by E. Meyer (*Ann. d. Phys.*, 71, 1923, p.

[1] See also a paper by W. Schottky (*Zeit. f. Phys.*, 36, 1926, p. 689) on the Law of Low [Frequency] Reception in Acoustics. This is a restatement of the Principle of Reciprocity; the principle dates from Helmholtz, and, as applied to sound fields, is given in Rayleigh, II, p. 145. Rayleigh applied it to the conical horn (II, p. 146); also to elucidate a certain experiment of Tyndall's. (See "The Application of the Principle of Reciprocity to Acoustics," *Proc. Roy. Soc.*, 25, 1876, p. 118, or "Scientific Papers," I, p. 305.) There are certain restrictions on its applicability, and indeed, in Tyndall's experiment it was found to be not applicable. Schottky's application is to the comparison between the receiving-efficiency and the radiating-efficiency of the "Band-Sprecher," a device noted in the next section.

567) deals with the force due to the impact of sound waves on resonant membranes. The vibrations of a disc clamped at the center are treated by R. V. Southwell, *Proc. Roy. Soc.*, A101, 1922, p. 133.

Loud Speaking Telephones

The typical loud speaker with a horn is an outgrowth of the telephone, and as might be expected, there are many current models of the device. The Western Electric model, which is representative, is described in *Electrician*, 84, Mar. 12, 1920, p. 300; *Teleg. and Tel. Age*, 40, July 16, 1922, p. 321; also in various bulletins, supplied by the Western Electric Company. Circuits for use with the apparatus on a large scale are described by Green and Maxfield, *Trans. A.I.E.E.*, 42, 1923, p. 64; with discussion later, p. 83; see also the paper by Martin and Clark, *Trans. A.I.E.E.*, 42, 1923, p. 75. The Marconi Loud Speaker and circuits for operating it are described by H. J. Round in *Wireless World*, XV, Dec. 17, 1924, p. 365.

To get away from horns, many devices have been proposed; most of these employ some sort of a light diaphragm, closely coupled to an electromagnetic driving element. Typical is the Western Electric model 540 AW, popularly known as the cone type, as the diaphragm is a large paper cone. (See *Sci. Am.*, Dec. 1924, p. 390; *Q.S.T.*, 8, Dec. 1924, p. 27; also Bulletin No. T-750 supplied by the Western Electric Company.)

The device of C. W. Hewlett, in which a large conducting diaphragm, placed in a strong magnetic field, is driven by the induced eddy currents, is described in *Jour. Opt. Soc. Am.*, 6, 1922, p. 1059, and its calibration is discussed in *Phys. Rev.*, 23, 1924, p. 310; the theory of the device is given in *Radio Broadcast*, 7, Aug. 1925, p. 508. The Gaumont loud speaker, in which thin driving coil is mounted on the diaphragm, the diaphragm being placed in a strong magnetic field, is described by Bonneau in *Bull. Soc. Franc. des Electriciens*, 4, 1924, p. 157; see also *le Génie Civil*, 84, 1924, p. 526. The German "Band-Sprecher" is described by E. Gerlach (*Phys. Zeit.*, 25, 1924, p. 675; *Zeit. f. Tech. Phys.*, 5, 1924, p. 576) and W. Schottky (in

Zeit f. Tech. Phys., 5, 1924, p. 574 or *Phys. Zeit.*, 25, 1924, p. 672), also in *Electr. Nachr. Tech.*, 2, 1925, p. 157, and the publications of the Siemens-Konzern, who manufacture the device. Interesting, but so far non-commercial devices are the Frenophone of S. G. Brown (*J.I.E.E.* 62, 1924, p. 283) and the Johnsen-Rahbek device (*J.I.E.E.* 61, 1923, p. 713; see also K. Rottgardt, *Zeit. für Tech. Phys.*, 2, 1921, p. 315; and *Jahrb. der Draht. Teleg.*, 19, 1922, p. 299). Some general information on loud speakers is given in a paper by Rice and Kellogg, *J.A.I.E.E.*, 44, 1925, p. 982 (discussion p. 1015 following) entitled "Notes on the Development of a New Type of Hornless Loud Speaker," with a description of an instrument of the piston type they have recently developed. Interesting discussions of various phases of the loud speaker problem are given in *J.I.E.E.*, 62, 1924, p. 265, by a number of members of the Institution of Electric Engineers and the London Physical Society; also reported in *Proc. London Phys. Soc.*, 36, 1924 p. 114 and p. 211. An excellent paper on the general theory of loud speakers is that of H. Riegger, *Wissensch. Veröffent. aus d. Siemens-Konzern*, III, No. 2, 1924, p. 67; a paper by Trendelenburg, *ibid.*, IV, No. 2, 1925, p. 200, describes a New Method for testing these instruments. A recent paper by C. R. Hanna (*Proc. I.R.E.*, 13, 1925, p. 437) deals with the design of the balanced-type element for driving the loud speaker.

The loud speaker in various forms is in a state of rapid development, and it is likely that instruments both with and without horns have possibilities that have not yet been fully exploited. Many designers of loud speaking apparatus all but ignore acoustical principles, in operating and testing their devices, but with the revival of interest in theoretical acoustics, and the dissemination of more reliable information as to the duty expected of the loud speaker, the more faulty apparatus is in process of elimination.

The Thermophone

This device, while inefficient as compared with a telephone receiver, has the advantage of being free from resonance. The

theory is given by Arnold and Crandall (*Phys. Rev.*, X, 1917, p. 22); also in a later paper by E. C. Wente (*ibid.*, XIX, 1922, p. 333) which precises the calibration of the instrument. The thermophone has also been applied by Wente to the calibration of the condenser transmitter.

Resonators; Hot Wire Microphones

Interesting recent work on resonators has been done by Hahnemann and Hecht (*Phys. Zeit.*, 21, 1920, p. 187; *ibid.*, 22, 1921, p. 353); also by A. T. Jones (*Phys. Rev.*, 25, 1925, p. 696 and p. 705). In this connection, the reader may be interested in the action of the Tucker Microphone, which consists of a hot wire placed in the mouth of a resonator. This is described by Tucker and Paris (*Phil. Trans. Roy. Soc.*, vol. A221, 1921, p. 389) followed by further work of A. T. Paris (*Proc. Roy Soc.* A101, 1922, p. 391; *Phil. Mag.*, 48, 1924, p. 769) on double resonators; there is a good paper by Paris on the magnification of acoustic vibrations by resonators in *Science Progress*, XX, No. 77, 1925, p. 68. The pin-hole resonator of C. Barus (*Proc. Nat. Acad. Sci.*, 8, 1922, p. 163) has been studied by P. E. Sabine, *Phys. Rev.*, 23, 1924, p. 116.

The hot wire microphone in itself is considered by A. V. Hippel, *Ann. d. Phys.*, 76, No. 6, 1925, p. 590; a thermo-couple instrument for Sound (Intensity) Measurements is described by W. Spaeth, *Zeit. f. Tech. Phys.*, 6, 1925, p. 372.

The Rayleigh Disc

It is of interest to note that Mallet and Dutton (*J.I.E.E.* 63, 1925, p. 502) make use of the Rayleigh Disc to measure the sound output of the telephone; thus Lord Rayleigh's device of 1882 (*Phil. Mag.*, XIV. p. 186, 1882, or "Papers," Vol. II, p. 132; see also "Sound," II, p. 44) is still in current use. It has the advantage of giving a steady deflection, proportional to the square of the particle velocity in the undisturbed field; the radiometer (§ 48) is the only other purely acoustic measuring instrument of which this can be said. The sensitiveness of the disc can be

increased by placing it at a velocity loop in a resonant chamber; under these conditions the device is most serviceable for measurements at a single frequency. Care must be taken not to make the disc so large that its size is comparable to the wave length of the sound; its anomalous behavior under these circumstances is treated by C. H. Skinner, *Phys. Rev.*, 27, 1926, p. 346.

The Phonodeik

This instrument, used by D. C. Miller in his analyses of speech and music, is described fully in his "The Science of Musical Sounds" to which reference has already been made (§ 1). A recent paper by S. H. Anderson (*J. Opt. Soc. Am.*, 11, 1925, p. 31) deals with design and calibration, and gives some further useful references.

The Condenser Transmitter

With microphones as such, the acoustic laboratory must proceed with caution, due to the fact that the sensitiveness of the microphonic substance is inherently hard to control. For precision work in sound detection over a range of frequencies electromagnetic or electrostatic devices which have been highly damped are much more reliable. Representative of the first class is the Marconi-Sykes Magnetophone, described by H. J. Round in the *Wireless World*, XV, Nov. 26, 1924, p. 260. The condenser transmitter of E. C. Wente is representative of instruments of the latter class; its more recent features, and the calibration of the device, are described by him in *Phys. Rev.*, XIX, 1922, p. 498. Information on recent models can be had from the Western Electric Co., Inc.

The Kondensator-Mikrophon of H. Riegger (described by F. Trendelenburg, *Wiss. Veröffent. d. Siemens-Konzern*, III, No. 2, 1924, p. 46) is quite different from Wente's instrument, both in its construction, and in the much less straightforward way in which it is used. It seems to have more complicated mechanical characteristics, and it is used in a high-frequency

circuit so that incident speech waves modulate high-frequency oscillations; a detailed argument, according to which the instrument functions as a faithful recorder of complex sound waves, is contained in the reference cited.

The condenser transmitter of Wente has been applied in many studies by the staff of the Bell Telephone Laboratories which required distortionless transmission of speech sounds. See for example, the paper by H. Fletcher on " The Nature of Speech and its Interpretation" (*Jour. Fr. Inst.*, 193, 1922, p. 729); the paper of Crandall and MacKenzie on " Energy Distribution in Speech" (*Phys. Rev.*, XIX, 1922, p. 221); also Crandall and Sacia " Dynamical Study of the Vowel Sounds," (*Bell System Tech. Jour.*, III, 1924, p. 232). In work of this kind a distortion-free amplifier circuit is necessary, and if records of sounds are to be made with the electrical circuit, the oscillograph used must also be distortionless. The oscillograph vibrator, whose general construction is explained by Kennelly (" Electrical Vibration Instruments," Ch. XV) must be highly damped and carefully calibrated; since it is a relatively simple vibrating system this is entirely practicable. A description of the amplifier used by Crandall and Sacia for making accurate records of vowel and consonant sounds appears in the paper by Crandall noted below. The important requirements for any amplifier used in connection with precision apparatus are to make the frequency response as nearly constant as possible over the working range, and to eliminate distortion due to overloading, for the maximum output furnished by the amplifier.

Speech and Hearing

The voice and the ear are, broadly speaking, the most important acoustical apparatus in use; and though many acoustical experiments may be made independently of speech and hearing, no worker in acoustics is likely to ignore these interesting phenomena, whatever his special interests may be. The references given here to recent work on speech and hearing do not discriminate between the fundamental study of these phenomena in themselves, and the more restricted work, which

relates to their bearing on other fields of acoustics, and to their utility in the laboratory.

Hearing has been extensively studied by members of the staff of the Bell Telephone Laboratories; a paper by H. Fletcher in *Bell System Tech. Jour.*, II, Oct. 1923, p. 145 (also in *Jour. Fr. Inst.*, 196, 1923, p. 289) deals with Physical Measurements of Audition and gives a very complete bibliography of the subject. The recent letter of R. L. Wegel in *Nature* (116, 1925, p. 393) gives a good statement of the present status of the resonance theory of hearing.[1] See also *Bell System Tech. Jour.*, IV, July, 1925, p. 375, for a compilation of the best available data on the Constants of Speech and Hearing. In this paper will be found references to the data of Fletcher and Steinberg, Fletcher and Wegel, and Wegel and Lane; also references to other recent work on various phases of hearing—all important if the ear is to be used as a sound-detecting instrument. The Audiometer, an instrument for measuring the sensitiveness of the ear, is described by Fletcher in the *Trans. Coll. of Physicians of Phila.*, 45, Series 3, 1923, p. 489; also in several bulletins furnished by the Western Electric Company. Audiometric Methods and their Applications are described by E. P. Fowler and R. L. Wegel in *Trans. Am. Laryng. Rhinol. and Otol. Soc.*, 1922, p. 98.

The experiments of F. W. Kranz of the Riverbank Laboratories on the sensitivity of the ear (*Phys. Rev.*, 21, 1923, p. 573, and *ibid.*, 22, 1923, p. 66) are being continued. The relative sensitivity at different levels of loudness is treated by D. MacKenzie in *Phys. Rev.*, 20, 1922, p. 331. V. O. Knudsen has a paper on the sensibility of the ear in *Phys. Rev.*, 21, 1923, p. 84; also a paper on the effect of tones and noise on speech reception by the ear, in *Phys. Rev.*, 26, 1925, p. 133. The relation between the loudness of a sound and its physical stimulus is treated by J. C. Steinberg, *Phys. Rev.*, 26, 1925, p. 507.

[1] The monograph by George Wilkinson and Albert A. Gray entitled "The Mechanism of the Cochlea" (London, 1924) is an interesting and well written "Restatement of the Resonance Theory of Hearing." Many of the questions which were still open when this volume was written have been answered by the more recent work noted, but the monograph is a distinct contribution to the literature of hearing.

A general study of important German contributions to the theory of hearing (and some of the American contributions already mentioned) is given by E. Waetzmann, *Phys. Zeit.*, XXVI, 1925, p. 740. Waetzmann also has a book on the "Resonance Theory of Hearing," Braunschweig, 1912.

In using the ear for detecting purposes it may be pointed out that in general, many more observations are required for a given degree of precision than if a mechanical or electrical instrument is used. (Cf. § 53, remarks on Sabine's technique.) The ear is at its best if two sounds of the same wave form and the same intensity are to be compared; the accuracy obtainable, of the order of five per cent for a single observation, is comparable to that obtainable with the eye in photometrical measurements.

The principal technical application of the Binaural Effect is to direction finding, as in submarine detection and airplane location. These subjects will be dealt with below.

The most original work on the speech sounds in recent years has been in synthesizing them, from the transient vibrations of electrical or mechanical resonators. (Most of the vowels, as is evident from their energy-frequency spectra, can be imitated by suitably exciting a double-resonator system; in such a system the external orifice corresponds to the mouth, the chamber behind it to the buccal cavity, and the inner chamber to the pharynx. The coupling between the two chambers may be varied by changing the size of the orifice between them; and since the volumes of the chambers themselves may be varied, a number of arrangements—of loose or close coupling—can be found for producing, with sufficient exactitude, a given doubly-resonant effect.) The analogous electrical method was used by J. Q. Stewart, reported in *Nature*, 110, Sept. 2, 1922, p. 311; the mechanical method by Sir Richard Paget (*Proc. Roy. Soc.*, A102, 1923, p. 752) for Vowel Sounds; and again by Paget for Consonant Sounds, *ibid.*, A106, 1924, p. 150; see also *Nature*, 111, Jan. 6, 1923; *Proc. Lond. Phys. Soc.*, 36, 1924, p. 213; and *J.I.E.E.*, 62, 1924, p. 963 (lecture deliv-

ered Mar. 20, 1924). In this connection the Artificial Larynx of Fletcher and Lane (a device used in connection with the mouth cavities, to produce speech when the natural larynx has been removed) is of interest. This is described in *Western Electric News*, Jan. 1925; *Philadelphia Record*, Jan. 18, 1925, and other news sources of that time; also in literature supplied by the Western Electric Company.

The structure of vowel sounds and the frequencies of certain consonant sounds are treated by C. Stumpf, *Ber. d. Preuss. Akad., Berlin*, 1918, p. 333; also *ibid.*, 1921, p. 636; a later paper by Stumpf appears in *Beitr. z. Anat., Physiol., Pathol., des Ohres*, etc., 17, 1921, p. 151; see also *ibid.*, p. 143, and p. 182. Another German investigation, by F. Trendelenburg, is given in *Wissensch. Veröffent. dem Siemens-Konzern*, III, Heft 2, 1924, p. 43, and *ibid.*, IV, Heft 1, 1925, p. 1; see also *E.T.Z.*, 46, 1925, p. 915; the first of these papers contains a bibliography of previous German work. A recent paper by Riegger and Trendelenburg, *Zeit. f. Tech. Phys.*, 7, 1926, p. 187, describes further studies in speech, with applications to the loud speaker problem.

Some recent work on speech at the Bell Laboratories has already been noted. The general problem of High Quality Reproduction of Speech is treated by Martin and Fletcher in *Trans. A.I.E.E.*, 43, 1924, p. 384. The work of Crandall and Sacia on Speech is being continued; a paper by Sacia on "Speech Power and Energy" and one by Crandall on (the wave forms of) "The Sounds of Speech" appear in *Bell System Tech. Jour.*, IV, Oct., 1925. The results of all the investigations on speech point to definite characteristic frequencies and energy distributions, both in time and frequency, for the individual sounds. (The semi-vowel sounds, for example, have frequency spectra which suggest systems of four degrees of freedom, as might be expected, with the naso-pharynx and nasal cavities brought into play.) For discussion of the mechanism of the various sounds the reader had best refer to the original sources cited, and the collateral references there to be found; and other papers on this subject may be expected in due course from the Bell Laboratories.

Sound Analyzing Devices

The analysis of wave forms by computation, or by graphical methods belongs to the mathematical laboratory, but sound-analyzing devices making use of tuned circuits or other physical means may be mentioned here. The apparatus of Wegel and Moore (*Bell System Tech. Jour.*, III, 1924, p. 299) for electrical analyses, and the photomechanical method of C. F. Sacia (*J. Opt. Soc. of Am.*, 9, 1924, p. 487) are representative of purely physical devices for analysing complex wave forms.

The hot-wire microphone (noted above) has been applied by A. Fage (*Proc. Roy. Soc.*, A107, 1925, p. 451) to analyze the vibrations of air screws.

Submarine Signalling Apparatus; Underwater Sound Detection, and Depth Finding

As a result of the war, interest has been focussed on sound-detecting apparatus generally and many applications have been made to Submarine Signalling, Acoustic Depth-finding, Submarine and Airplane Detection, Sound-Ranging, and Geophones.

Reference has already been made to two general sources of information on submarine signalling. The older art is represented by several papers noted in section II of " Certain Problems in Acoustics" (*Bull. Nat. Res. Council, loc. cit.*); more recent work described in C. V. Drysdale's Chapter (IX) of the collective work " Mechanical Properties of Fluids," and in his Eleventh Kelvin Lecture, *J.I.E.E.*, 58, 1920, p. 572. Another source of general information is the " Unterwasserschalltechnik" of F. Aigner, Berlin (Krayn) 1922.

The theory of receiving sound in water is dealt with by H. A. Wilson (*Phys. Rev.*, XV, 1920, p. 178). The reaction of the medium (water) on elastic plates is the subject of a paper by H. Lamb (*Proc. Roy. Soc.*, A98, 1920, p. 205). The mechanics of diaphragms immersed in liquids is also treated by Powell and Roberts in *Proc. London Phys. Soc.*, 35, 1923, p. 170, and by J. H. Powell, *ibid.*, 37, 1925, p. 84; these writers confirm Lamb's

theoretical work. Diaphragms capable of continuous tuning are described by L. V. King in *Proc. Roy. Soc.*, A99, 1921, p. 163. A paper by Barlow and Keene (*Phil. Trans.* A222, 1922, p. 131) deals with the analysis of sounds in water. Two papers by H. Hecht on underwater acoustics will be found in *Zeit. für Tech. Phys.*, 2, 1921, p. 265 and p. 337. A paper by H. Lichte (*Phys. Zeit.*, 20, 1919, p. 385) discusses the effect of a temperature gradient in water on the signalling range.

Explosions under water are treated by H. Lamb (*Phil. Mag.* 45, 1923, p. 257) and by Ramsauer (*Ann. d. Phys.*, 72, 1923, p. 265). There is a paper on Cavitation in the Propagation of Sound, by R. W. Boyle, *Proc. and Trans.* of *Roy. Soc. Canada*, 16, Series III, 1922, p. 157.

The publications of H. C. Hayes summarize many important features of American practice in underwater sound detection. There are two articles in Vol. LIX, *Proc. Am. Phil. Soc.*, 1920; No. 1, p. 1, and No. 5, p. 371. The first of these articles contains, for example, descriptions of the Compensator and several arrangements of multiple unit acoustic receivers used with it, which were described by M. Mason in an unpublished paper before the *Am. Phys. Soc.*, at the Washington meeting, April, 1919; see also British Patent No. 146,192, Dec. 28, 1921, and U. S. Patent No. 1,422,876, July 18, 1922. The electrical compensator of G. W. Pierce is described in § 287 of his book on "Electric Oscillations and Electric Waves;" see also British Patent No. 146,163, Oct. 25, 1921, and *Proc. Am. Acad.*, 57, No. 8, May 1922. The Western Electric Seaphone (the prototype of the rubber-diaphragm hydrophone, with inertia microphone attached) is covered by U. S. Patent No. 1,581,334.

Hayes' second article deals with the hydrophone as an aid to navigation. An article by Hayes in *Jour. Fr. Inst.*, 197, 1924, p. 323, deals with measuring ocean depths by Acoustic Methods; see also his article on Sonic Depth Finding, *Proc. Am. Phil. Soc.*, LXIII, 1924, p. 134. Other papers by H. C. Hayes will be found in the *Marine Review*, 51, 1921, p. 404; p. 466; p. 493. The work of L. V. King on Sounding is given in *Nature*, 114, 1924, p. 122; see also *ibid.*, 113, 1924, p. 463.

The work of H. C. Hayes on Sounding is also described in the *Hydrographic Review* (*Monaco*) 2, No. 1, 1924, p. 93. (This issue has also been referred to in connection with the work of Langevin and Chilowsky on High-Frequency Signalling (§ 41).) Other French work is described by F. Collin in *le Génie Civil*, 86, 1925, p. 38, and p. 64. A method of echo sounding developed by Behm at Kiel is described in *Bull. Technique du Bureau Veritas*, 6, 1924, p. 161.

The methods used by the Submarine Signal Co. (Boston) are described by H. V. Hayes in the *Engineer*, 129, 1920, p. 491, and in literature available from that concern.

British practice in sound detection centers about the work of Sir William Bragg and his associates; see *Engineering*, 107, 1919, p. 776, or *Nature*, 103, 1919, p. 467. Wood and Young treat the directional phase of sound detection under water in two papers: *Proc. Royal Soc.*, A100, 1921, p. 252 and p. 261. Other papers of interest are by J. C. McLennan, *Trans. North-East Coast Inst. of Engrs. and Shipbuilders*, 35, 1919, p. 386; and by du Bois-Reymond, *Zeit. f. Tech. Phys.*, 2, 1921, p. 234; see also the *London Times Engineering Supplement*, July, 1919, p. 220, and *Nature*, 104, 1919, p. 28 (Inaugural Address of Sir Charles A. Parsons).

Important work done by the French on Sound Detection is described by A. Troller, *La Nature*, 49, 1921, p. 4; see also *le Génie Civil*, 79, 1921, p. 375, p. 393 and p. 417; also A. Marcelin, *Bulletin de Recherches et Inventions*, Nos. 10, 11, 12, 1920, p. 513, p. 577, and p. 641; also H. Brillié, *le Génie Civil*, 80, 1922, p. 378, p. 397, and p. 427. Brillié's earlier papers (1919) in *le Génie Civil* have already been noted in § 31.

The work of the Signal Gesellschaft Laboratories at Kiel is described in "Submarine Acoustic Signalling Apparatus," by W. Hahnemann, *Proc. Inst. Rad. Eng.*, 11, 1923, p. 9. A good description of the large sound generator is given by du Bois-Reymond, Hahnemann and Hecht in *Zeit. f. Tech. Phys.*, II, 1921, p. 1 and 33; see also H. Gerdien, *Phys. Zeit.*, 22, 1921, p. 679. Hahnemann and Lichte describe the development of submarine detectors in Germany during the war, in

Die Naturwissenschaften, 8, 1920, p. 871. The natural oscillations of microphones for underwater detection are treated by P. Ludewig, *Phys. Zeit.*, 21, 1920, p. 305. Barkhausen and Lichte (*Ann. d. Phys.*, 62, 1920, p. 485) give the results of a comprehensive study of underwater transmission conditions; Aigner has a paper in *Zeit f. Phys.*, I, 1920, p. 161 on the most economical operating frequency. Many more references to German practice and to the earlier art, are available in the book of Aigner, cited above; and finally a résumé of signalling devices is given by W. Wolf, *Zeit. f. Fernmeldetechnik*, 2, 1921, p. 81.

A very good bibliography, particularly as to patents, on subaqueous signalling of all kinds is given in one of the abstracts of miscellaneous methods of communication published by *The Radio Review*, II, Sept., 1921, p. 487. (A number of significant patents in this field were granted by the British Patent Office at about that time.) Here will be found references to the war-time inventions of P. Langevin, M. I. Pupin, H. C. Hayes, R. L. Williams, R. A. Fessenden, M. Mason, G. W. Pierce, and the members of the staff of the Signalgesellschaft at Kiel, some of which have already been noted. Another good bibliography is given in a very complete article by M. Tenani on the general problem of sound detection, in *Rivista Marittima*, 57, 1924, p. 319. The field under review is wide, and embraces a good deal besides acoustics.

Radio-Acoustic Signalling

A combination of radio with acoustic signalling has been devised for the purpose of position finding at sea. In one typical arrangement, the acoustic signals are sinusoidal vibrations; an embodiment of this scheme is seen in British Patent No. 146,125, 1920, to R. L. Williams of the Submarine Signal Company. Many others might also be mentioned; the field is not new, and has been developed with increased impetus recently, due to advances in the radio art, and the acoustic developments resulting from the war. In another arrangement, a bomb is used to produce the acoustic signal; representative of this class

is the joint work of the U. S. Bureau of Standards and the U. S. Coast and Geodetic Survey, described by S. R. Winters in *Radio*, 6, July, 1924, p. 10; see also *Special Publication*, No. 107, 1924, *of the U. S. Coast and Geodetic Survey:* "Radio-Acoustic Method of Position Finding" by Heck, Eckhardt and Keiser.

Sound Ranging Apparatus

Reference has been made above to the Tucker microphone, which was used as a detector for sound ranging, that is, locating the direction and distance of cannon and projectiles. Other references to British practice will be found in Drysdale's Kelvin Lecture, cited above, and C. A. Parsons' address, *Nature*, 104, 1919, p. 28; see also p. 313 of "Mechanical Properties of Fluids," and *Nature* 104, 1919, p. 278. The American practice in reducing observations is described by Col. A. Trowbridge, *Jour. Fr. Inst.*, 189, 1920, p. 133; the work of the Army Engineers in collaboration with the Western Electric Company on apparatus and methods is described by E. B. Stephenson in *U. S. Engineer School, Occasional Papers*, No. 63, 1920, on Sound Ranging; see also J. B. Cress, *Military Engr.*, 12, 1920, p. 275. This work was based on British practice (the Bull-Tucker System). Certain theoretical problems are discussed by E. Eschlangon, *Rev. Scientifique*, 59, 1921, p. 164, also by H. W. Hodgkins in *Jour. U. S. Artillery*, 52, 1920, p. 41. Subaqueous sound ranging is treated by F. E. Smith in the *Engineer* 138, 1924, p. 534; also by H. C. Allen, *Coast Artillery Jour.*, 59, 1923, p. 35; see also *J.U.S. Artillery*, 54, 1921, p. 69; also, the article by R. B. Webb, *Coast Artillery Jour.*, 59, 1923, p. 17.

Direction Finding; Airplane Location

The subject of binaural hearing has necessarily entered the discussion given in several papers cited under Sound Ranging and Submarine Detection; but its most outstanding application is to the problem of locating the direction of a sound

source in air. Outstanding in the field of binaural hearing are the contributions of R. V. L. Hartley (*Phys. Rev.*, XIII, 1919, p. 373), G. W. Stewart (*Phys. Rev.*, XV, 1920, p. 425 and 432), and Hartley and Fry (*Phys. Rev.*, XVIII, 1921, p. 431); see also the bibliography (already cited) by Fletcher in *Jour. Frank. Inst.*, Sept., 1923. Some further references are available in a paper by E. Meyer (*E.T.Z.*, 46, 1925, p. 805) on "Stereo-acoustic Hearing." A recent paper by C. E. Lane (*Phys. Rev.*, 26, 1925, p. 401) on Binaural Beats is also of interest in connection with this subject.

The effect of irregularities in the air on sound transmission is considered by G. W. Stewart, *Phys. Rev.*, XIV, 1919, p. 376; his work on Aircraft Location is described in the same volume, p. 166. There is a paper on Aircraft Location by E. Waetzmann, in *Zeit. f. Techn. Phys.* 2, 1921, p. 191. In vol. 4, 1923, p. 99, *ibid.*, the paper by E. Lübcke describes water-tight apparatus for submarine listening to aircraft sounds. The analysis of aircraft sounds is treated by E. Waetzmann, *ibid.*, 2, 1921, p. 166, and by L. Prandtl, *ibid.*, 2, 1921, p. 244. There is a paper by W. S. Tucker on Sound Reception as Applied to Air Defence in *Jour. Roy Aeronaut. Soc.*, 28, 1924, p. 504.

Geophones

The geophone was developed originally in France for listening to the sounds sent through the ground, as the result of military mining operations. Since the war, mining engineers have been interested in the device, but few references are available to original work. An excellent general account of Geophones is given by Alan Leighton, *U. S. Bureau of Mines, Technical Paper* No. 277, 1922. A brief note by Ackley and Ralph in *Jour. Fr. Inst.*, 198, 1924, p. 711 or p. 834, on Improvements in Geophones by the use of Electrical Sound Amplifiers is followed by a more detailed account in a publication of the *U. S. Bureau of Mines*, *Report of Investigation* No. 2639. There is an excellent paper by H. S. Ball in *Jour. Inst. of Min. and Met.*, 28, 1919, p. 189, which contains material on the Geophone and other devices.

Addenda: Measurement of Absorbing Properties of Materials

There has recently come to hand a paper by Eckhardt and Chrisler (*U. S. Bureau of Standards*, Sci. Paper No. 526, April 28, 1926) on Transmission and Absorption of Sound; this is a continuation of the work at the Bureau (Sci. Paper No. 506) noted in Chap V. In the more recent paper a number of data are given on transmission and absorption by panels of various materials; and of particular interest is the method of determining absorption coefficients, which is based on that of H. O. Taylor to which reference has already been made (p. 109). In Eckhardt's method, the length of the tube is *fixed* (for resonance at a given frequency) and the reflecting power of the layer is given in terms of the pressure maxima and minima in the standing wave system, which are determined by an exploring tube. A correction is also made for the attenuation in the tube (Helmholtz effect) if this is required.

Some further notes on Wente's experiment may be of interest here, following the brief outline given on p. 109. The velocity of the driving piston is kept virtually constant, and the pressure near the piston (as determined by the exploring tube) thus measures the driving-point impedance of the apparatus. If p_1 represents the maximum driving point pressure (corresponding to an effective length of tube of an integral number of half wave lengths) and p_2 the minimum driving point pressure (the length of tube now differing by a quarter wave length from that of the first adjustment) the amplitude reflection coefficient of the layer at the end of the tube is

$$|r| = \frac{\sqrt{p_1/p_2} - 1}{\sqrt{p_1/p_2} + 1}$$

from which the absorption coefficient of the layer can be calculated (cf. problem 48, p. 227).

A more complete account of Wente's analysis and of his determination of the absorbing properties of materials, by this method, is in preparation.

INDEX OF NAMES

[*The numbers refer to pages*]

INDEX OF SUBJECTS

[The numbers refer to pages]

ERRATA

PAGE vi, *line* 11: *read* in *for* on.

PAGE 22, *equation* 24: *read* $\dfrac{\partial^2}{\partial r^2}$ *for* $\dfrac{\partial}{\partial^2 r}$.

PAGE 24, *equation* 24*a*: *read* $\dfrac{\partial^2}{\partial r^2}$ *for* $\dfrac{\partial}{\partial r^2}$.

PAGE 31, *equation* 38: *read* $\dfrac{\partial^2 p_2}{\partial r^2}$ *for* $\dfrac{\partial^2 p_2}{\partial r}$.

PAGE 42, *line* 5: *read* the case *for* case.

PAGE 47, *line* 1: *read* $\Psi_0 e^{i\omega t}$ *for* $\Psi e_0{}^{i\omega t}$.

PAGE 50, *line* 3: *read* include in *for* include.

PAGE 58, *line* 9 *from bottom*: *read* § 25 *for* § 15.

PAGE 82, *equation* 99: *read* $J_1(\alpha_k a)$ *for* $J_1(a_k r)$.

PAGE 92, *equation* 116: *read* $\left|\dfrac{dW}{dt}\right|_{\text{av.}}$ *for* $\left|\dfrac{dW}{dt}\right|$.

PAGE 100, *second footnote*, *line 1*:

insert following if, $Z_0 = 0$ and.

PAGE 115, *equation* 148: *read* $\displaystyle\int_0^t$ *for* $\displaystyle\int_0$;

equation 150: *read* $\dfrac{\partial\dot{\eta}}{\partial y}$ *for* $\dfrac{\partial\eta}{\partial y}$.

PAGE 116, *equation* 151*a*: *read* $\ddot{\phi}$ *for first* ϕ;

line 9: *read* $\mathbf{s}$ *for* s.

PAGE 137, *line* 10: *delete* " (" *before* 160).

CHAPTER IV, *read* $\left|\dfrac{dW}{dt}\right|_{\text{av.}}$ *for* $\dfrac{dW}{dt}$ *or* $\left|\dfrac{dW}{dt}\right|$.

PAGE 146, *line* 6: *read* $\displaystyle\int_S$ *for* $\displaystyle\int^S$.

PAGE 171, *line* 11: *read* a pulsating *for* pulsating.

PAGE 194, *equation* 120*a*: *read* $\mathbf{t}_{12}\mathbf{t}_{21}$ *for* $\mathbf{t}_{21}\mathbf{t}_{21}$.